A General Framework for Registered Functional Encryption via User-Specific Pre-Constraining

Tapas Pal[1] **Robert Schädlich**[2]

November 6, 2025

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Functional Encryption (FE) [TCC:BSW11]

$\mathsf{Setup}(1^\lambda) \to (\mathsf{mpk}, \mathsf{msk})$

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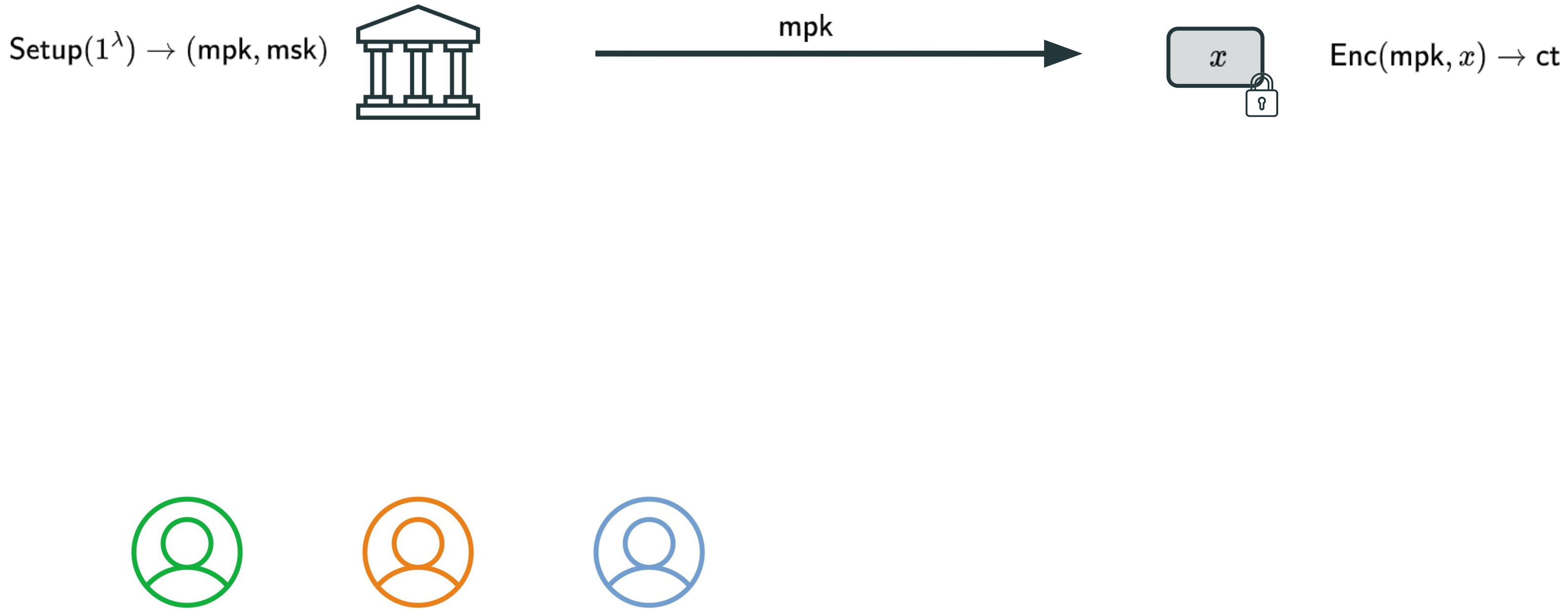

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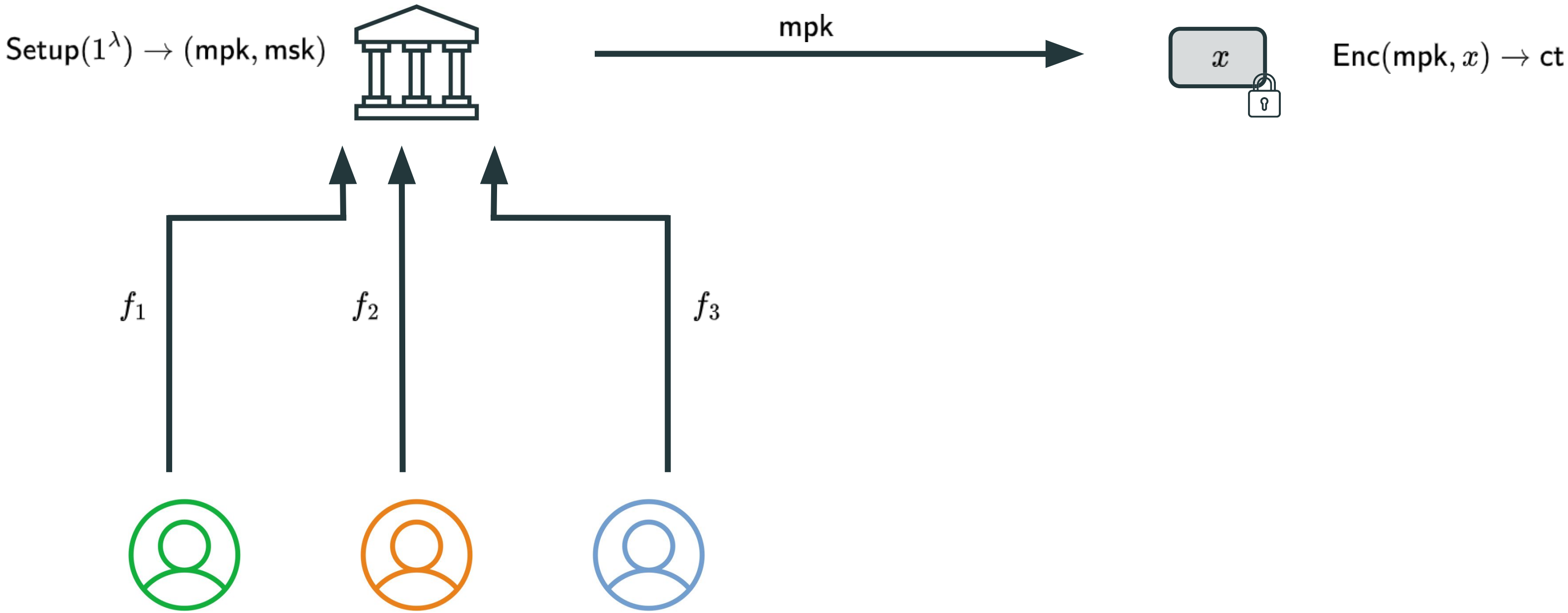

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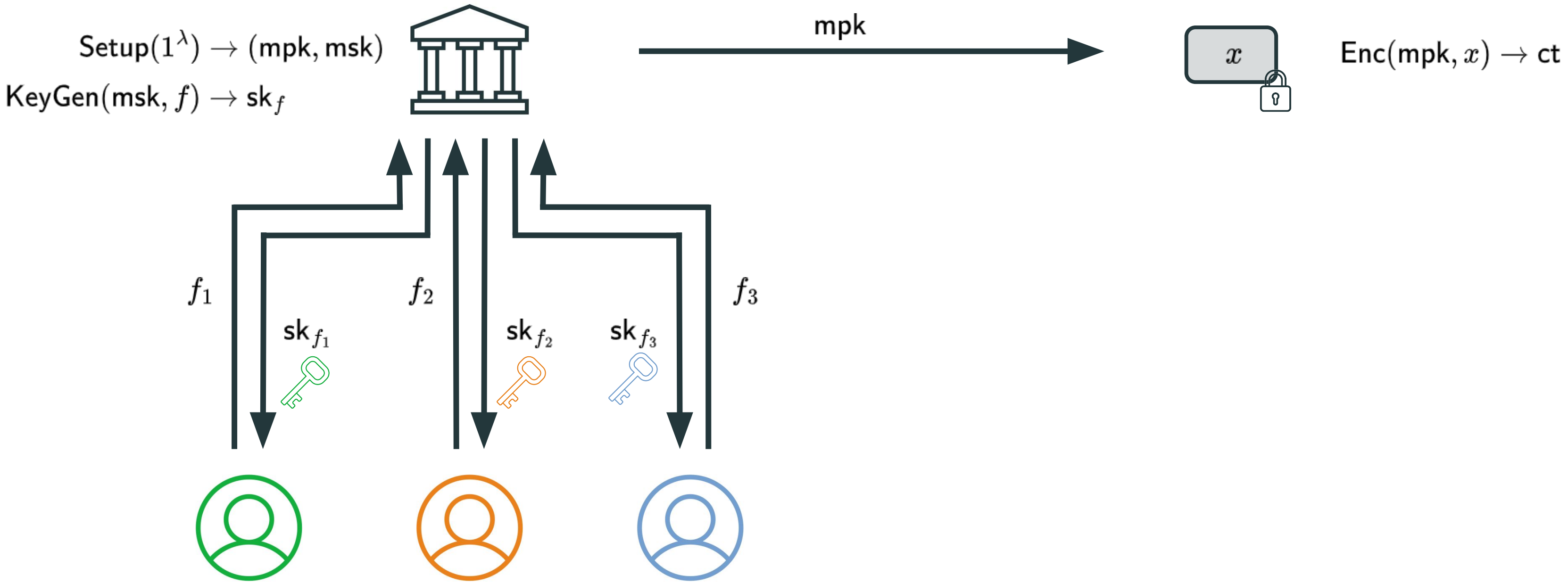

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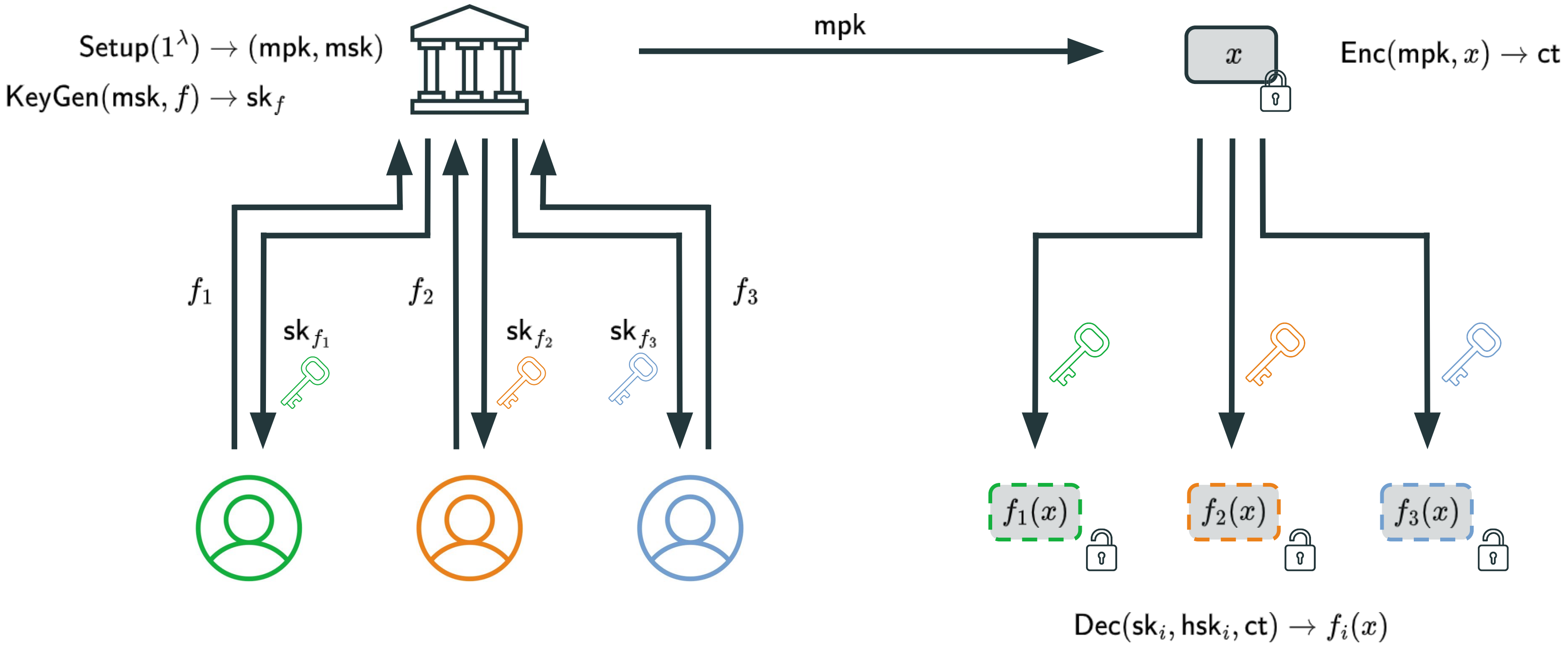

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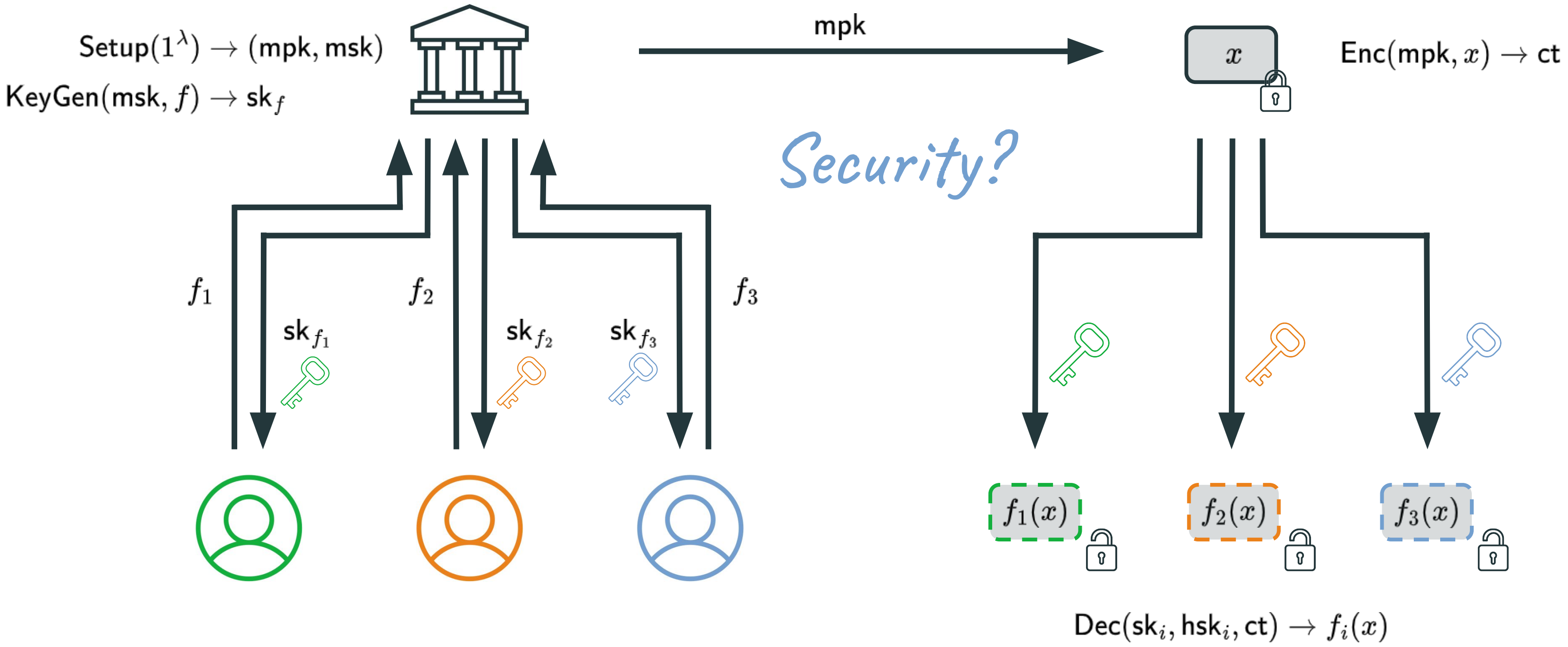

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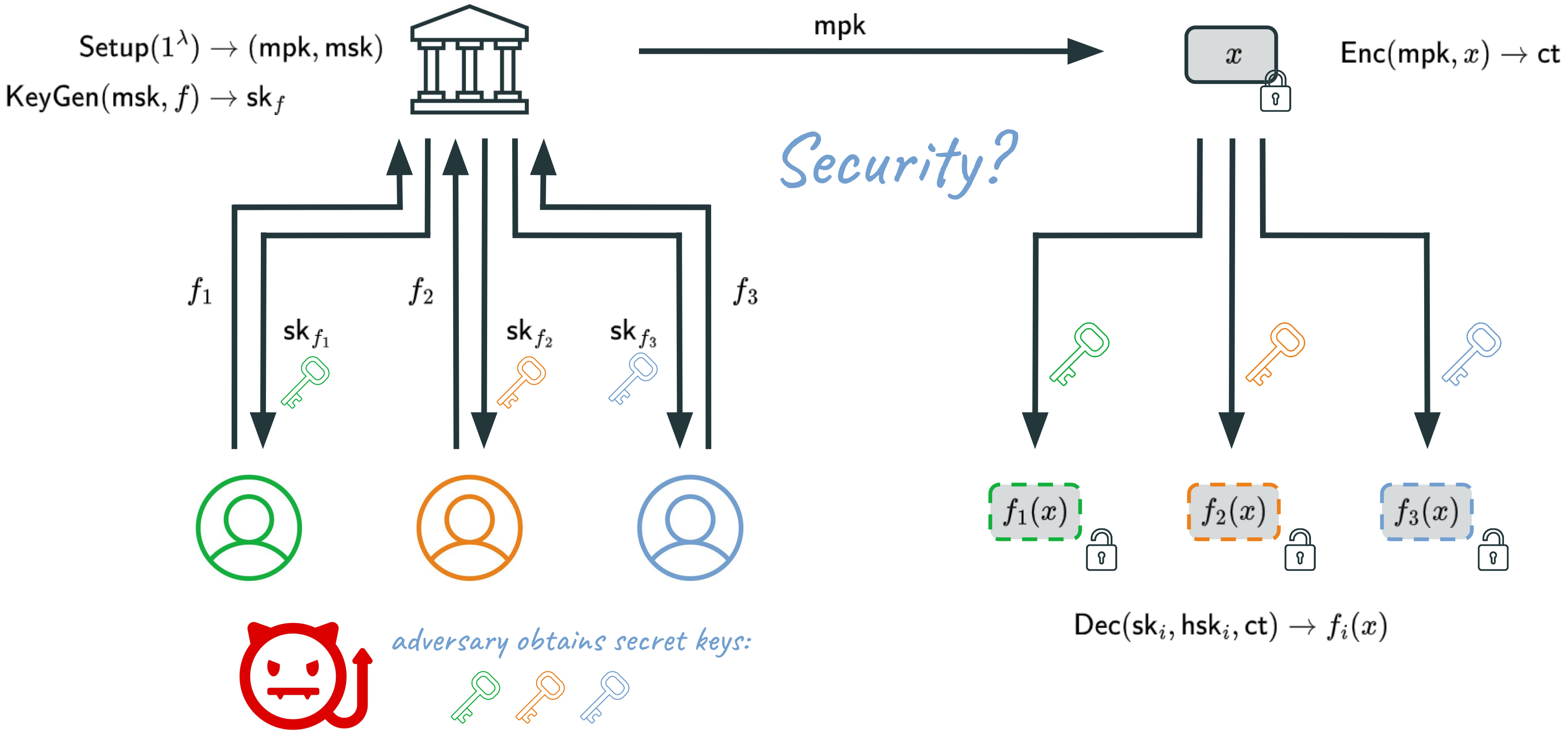

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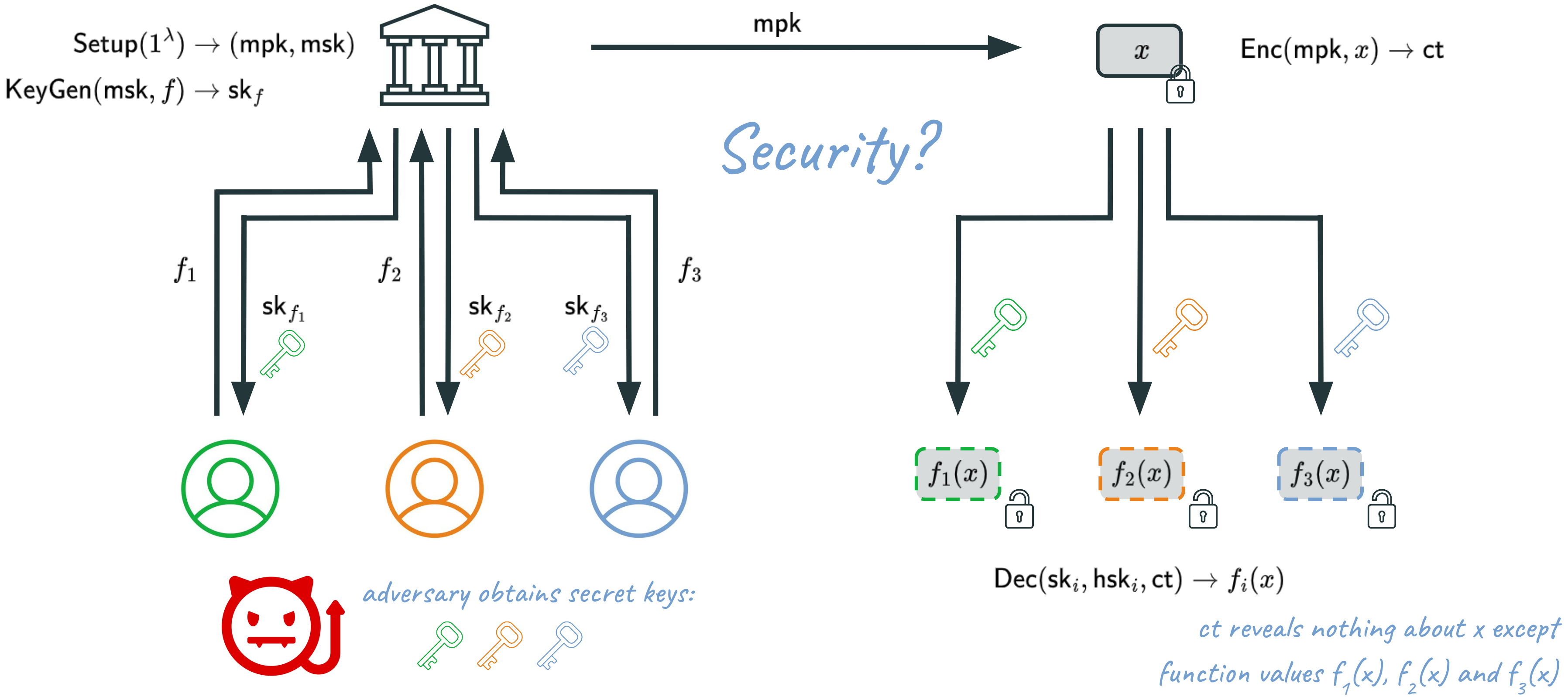

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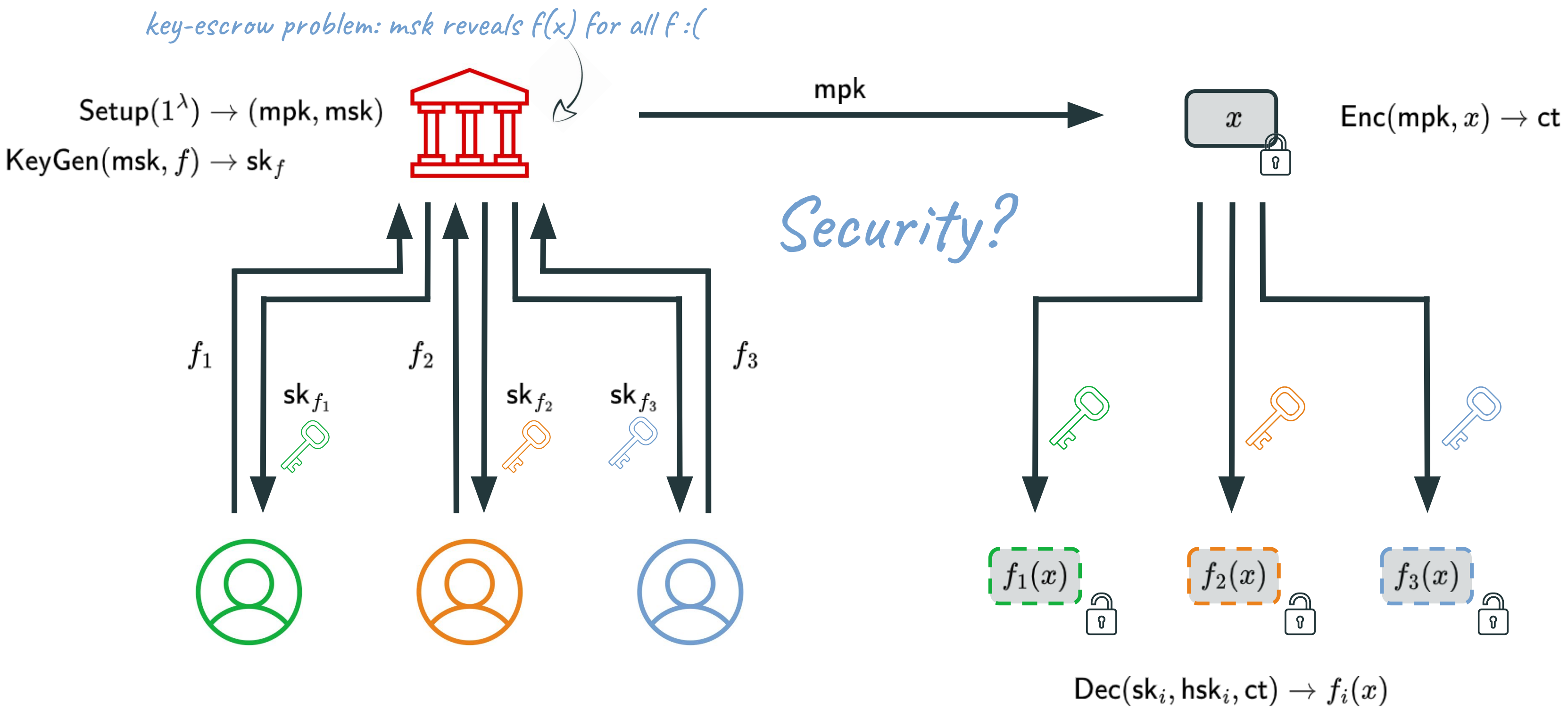

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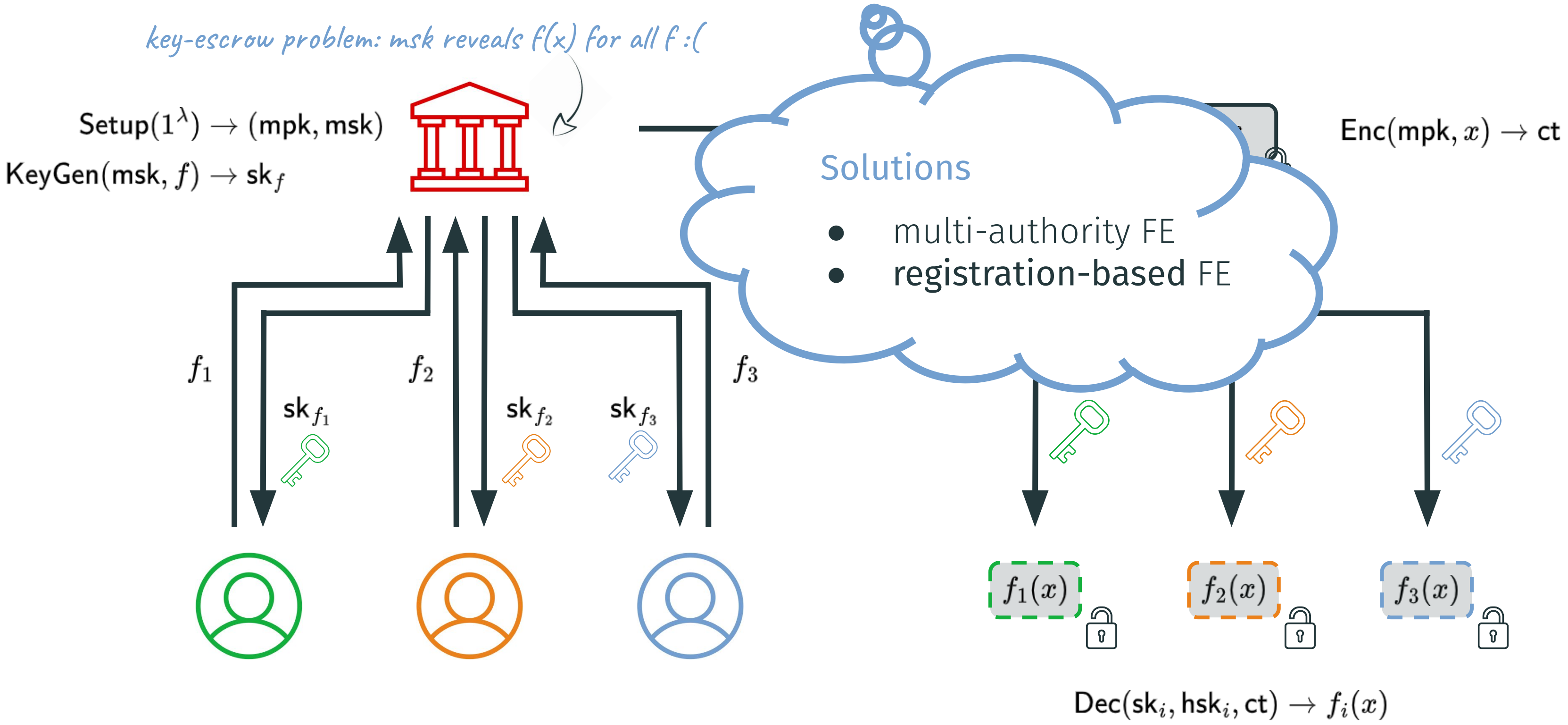
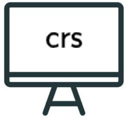


Solutions

- multi-authority FE
- **registration-based** FE

Registered Functional Encryption (RFE) [AC:FFM+23]

$\mathsf{Setup}(1^\lambda) \rightarrow \mathsf{crs}$



$\mathsf{pk}_1, \mathsf{sk}_1$ $\quad$ $\mathsf{pk}_2, \mathsf{sk}_2$ $\quad$ $\mathsf{pk}_3, \mathsf{sk}_3$

$\mathsf{KeyGen}(\mathsf{crs}, i) \rightarrow (\mathsf{pk}_i, \mathsf{sk}_i)$

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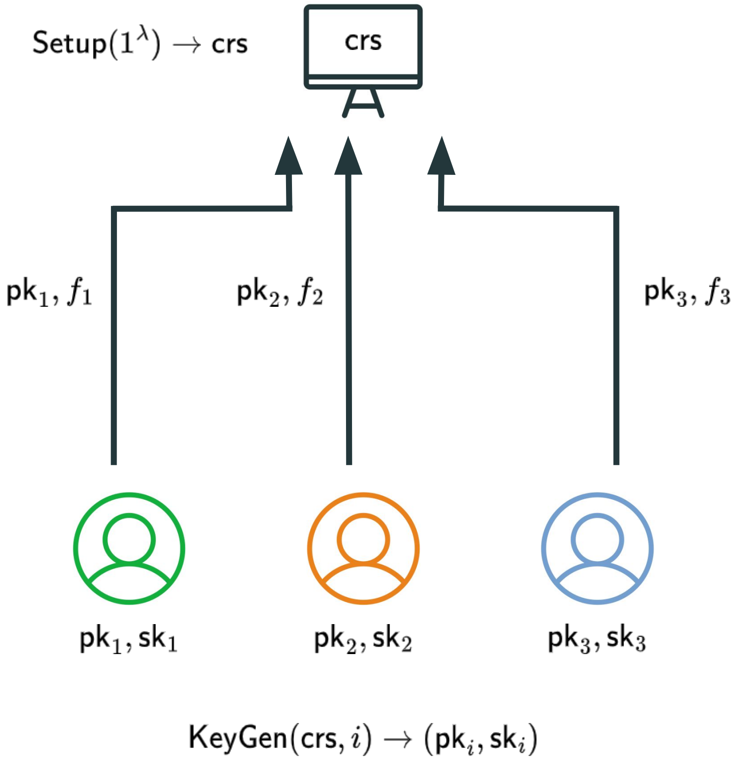

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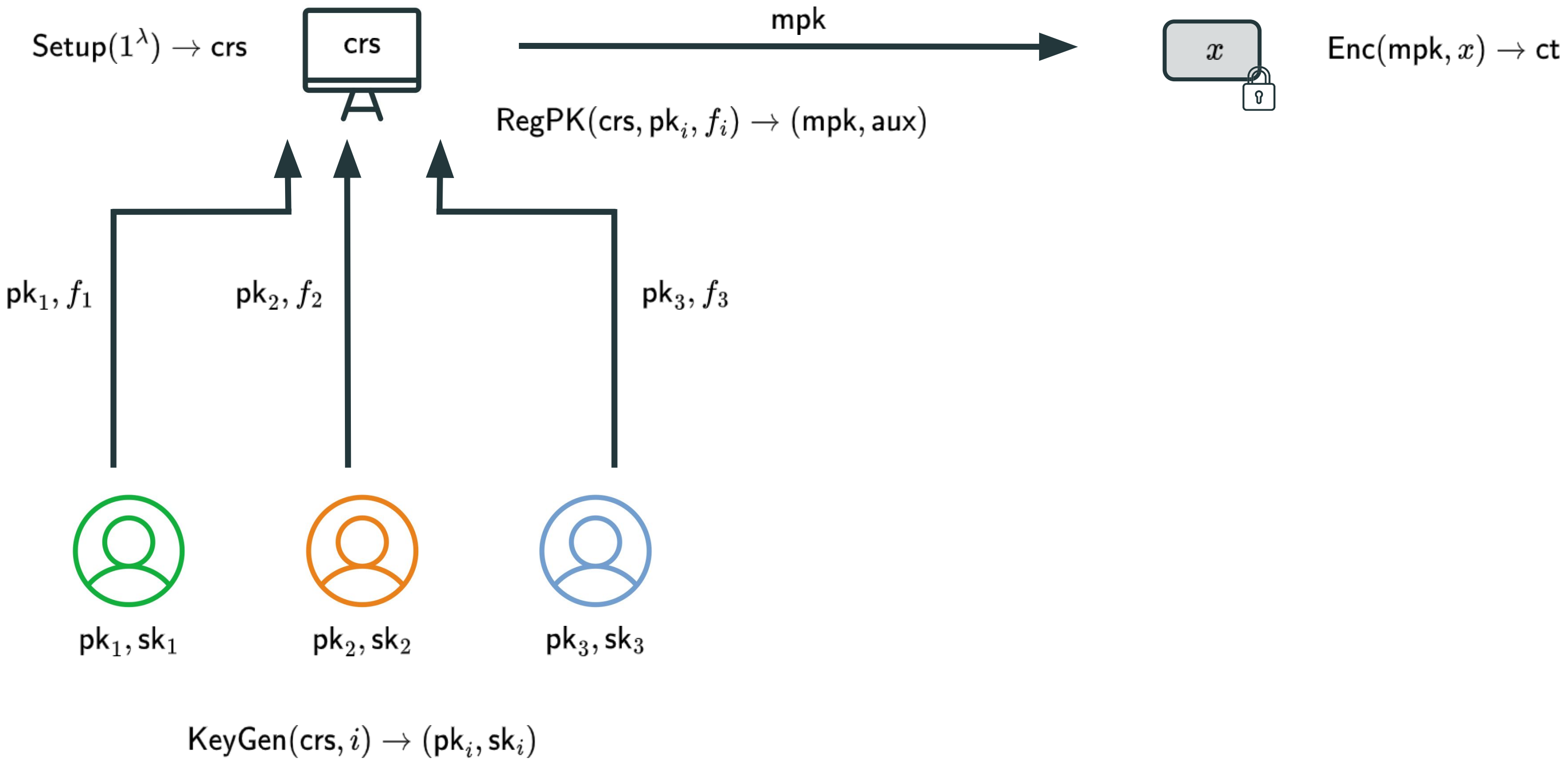

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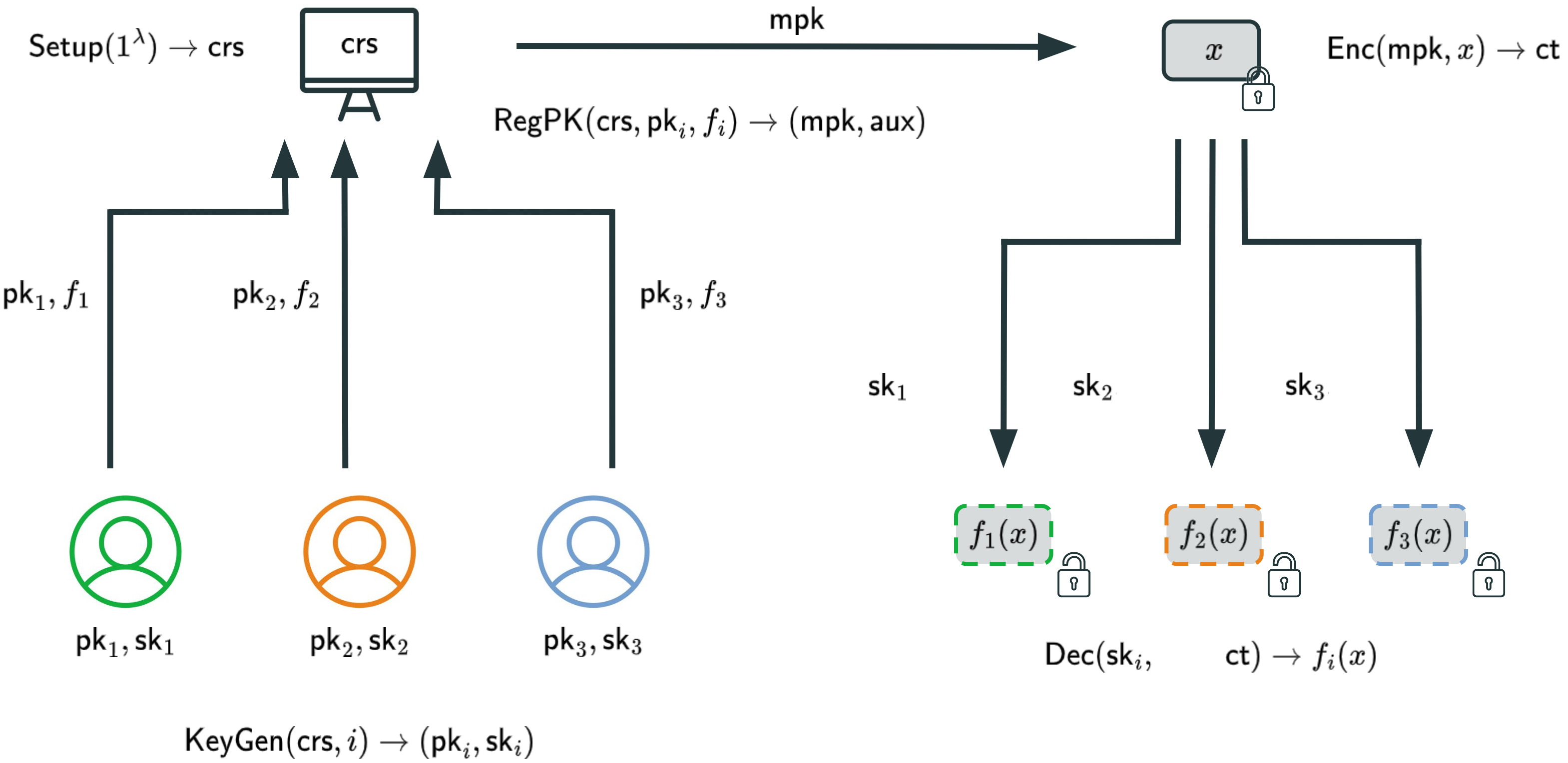

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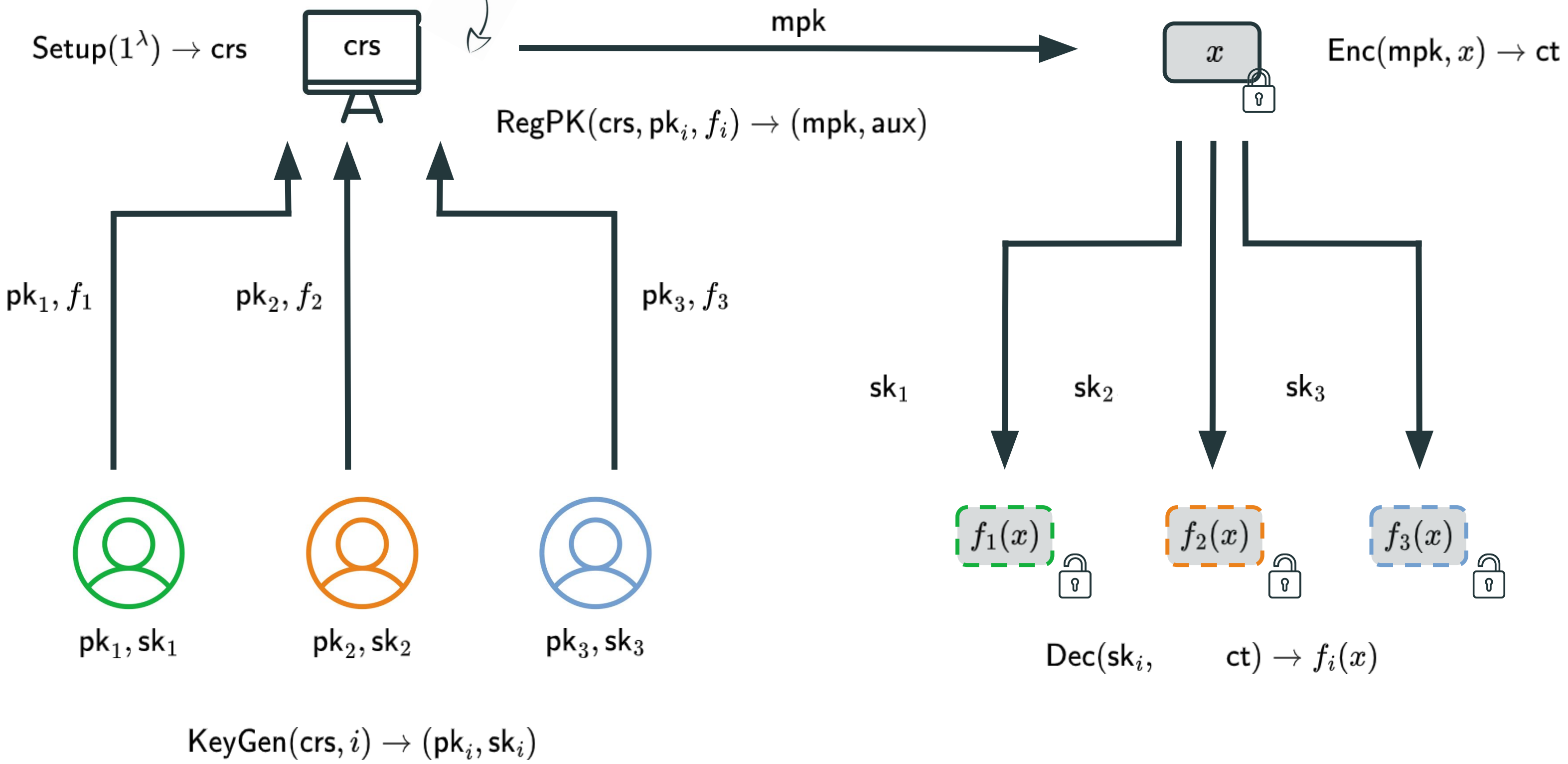
key curator is deterministic & holds no secret => key-escrow problem resolved!



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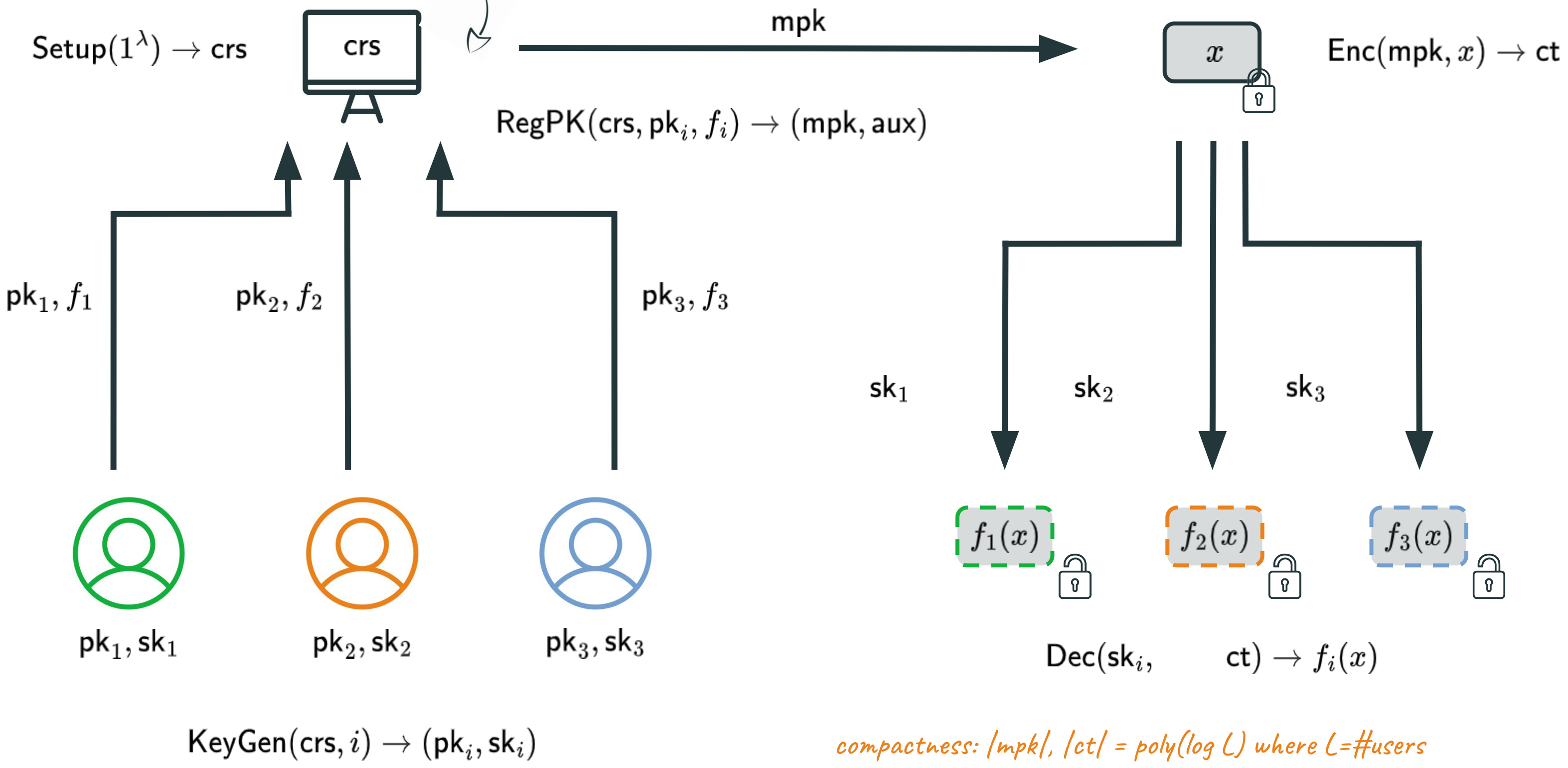

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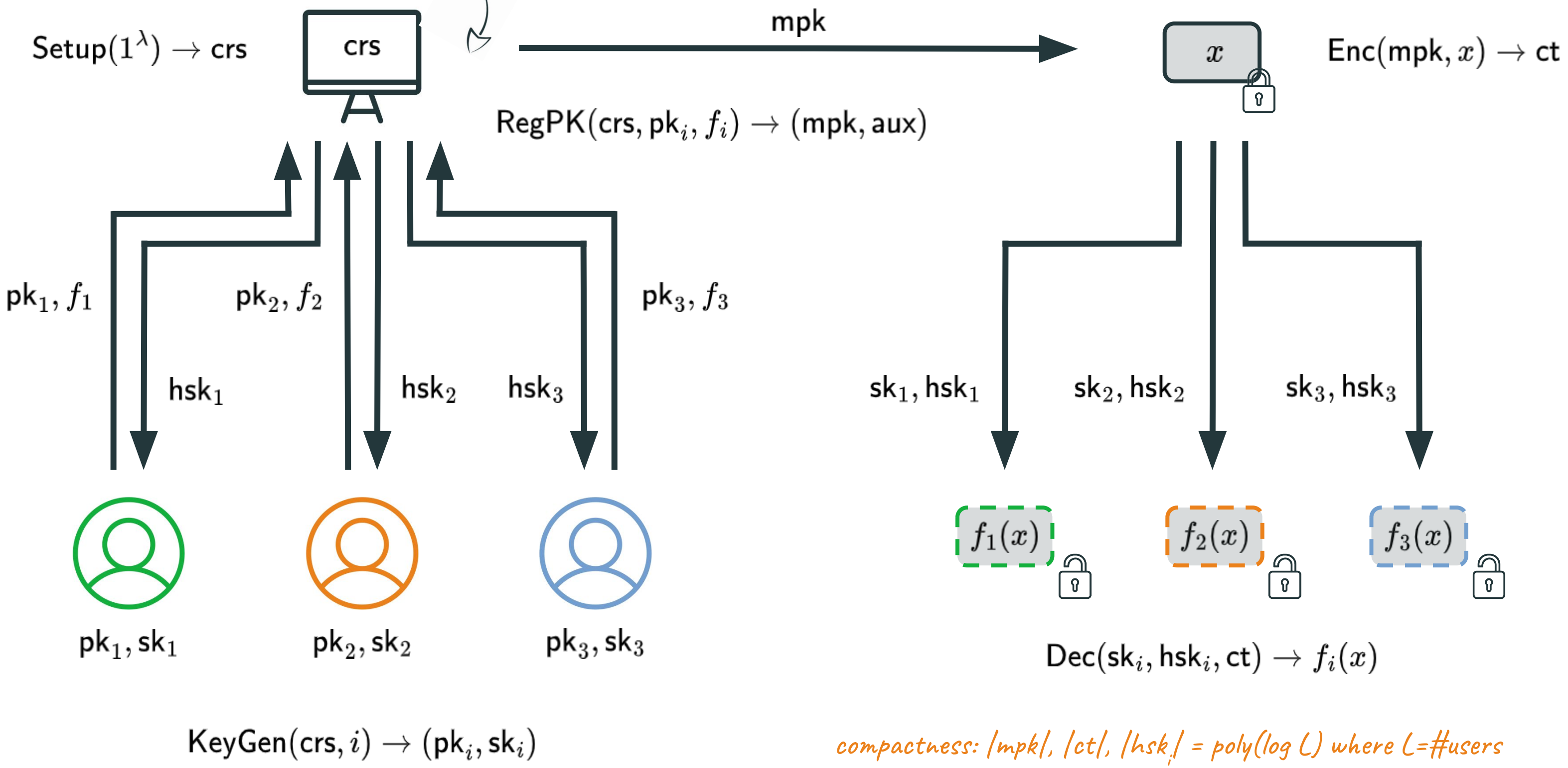

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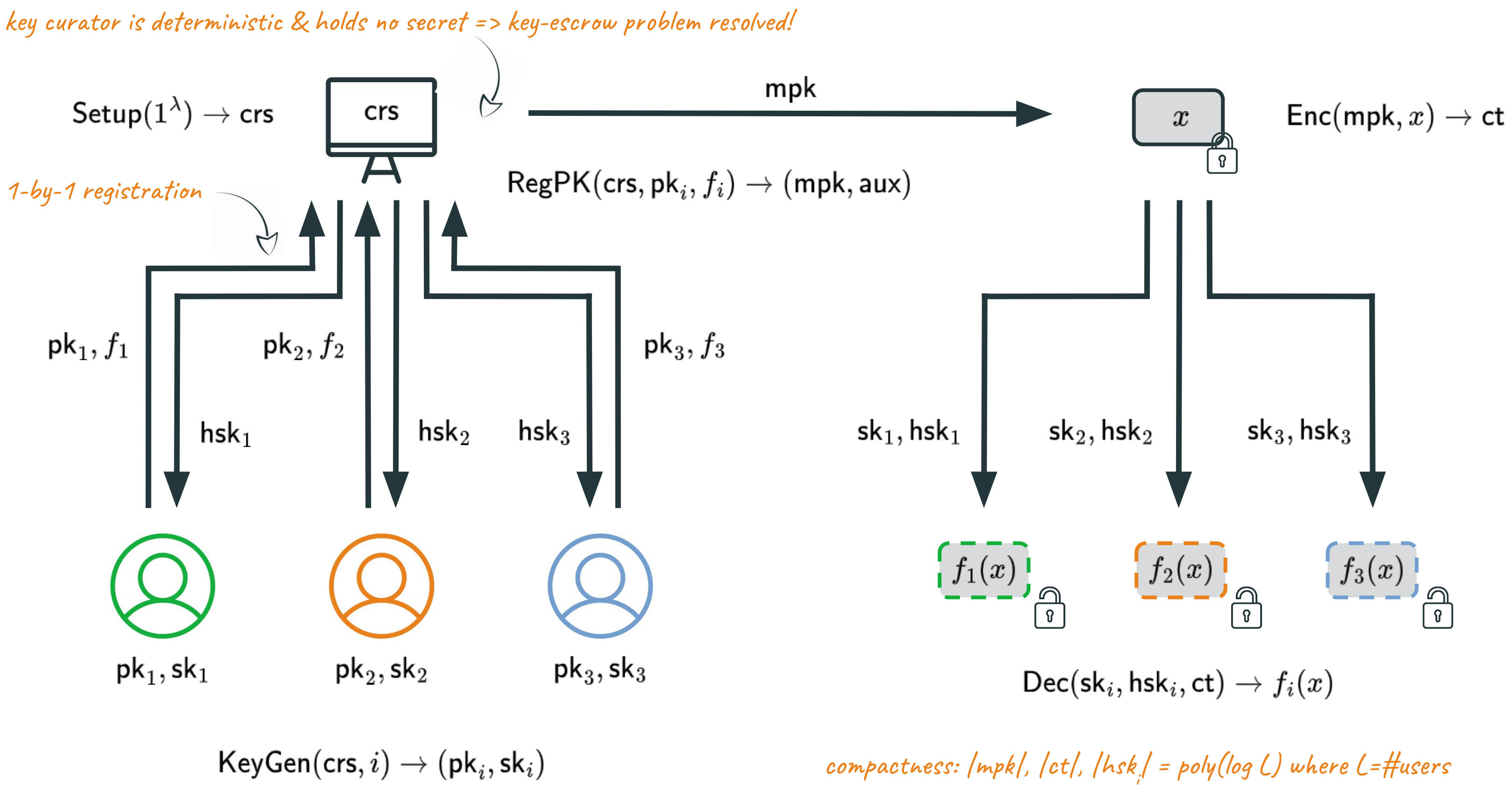

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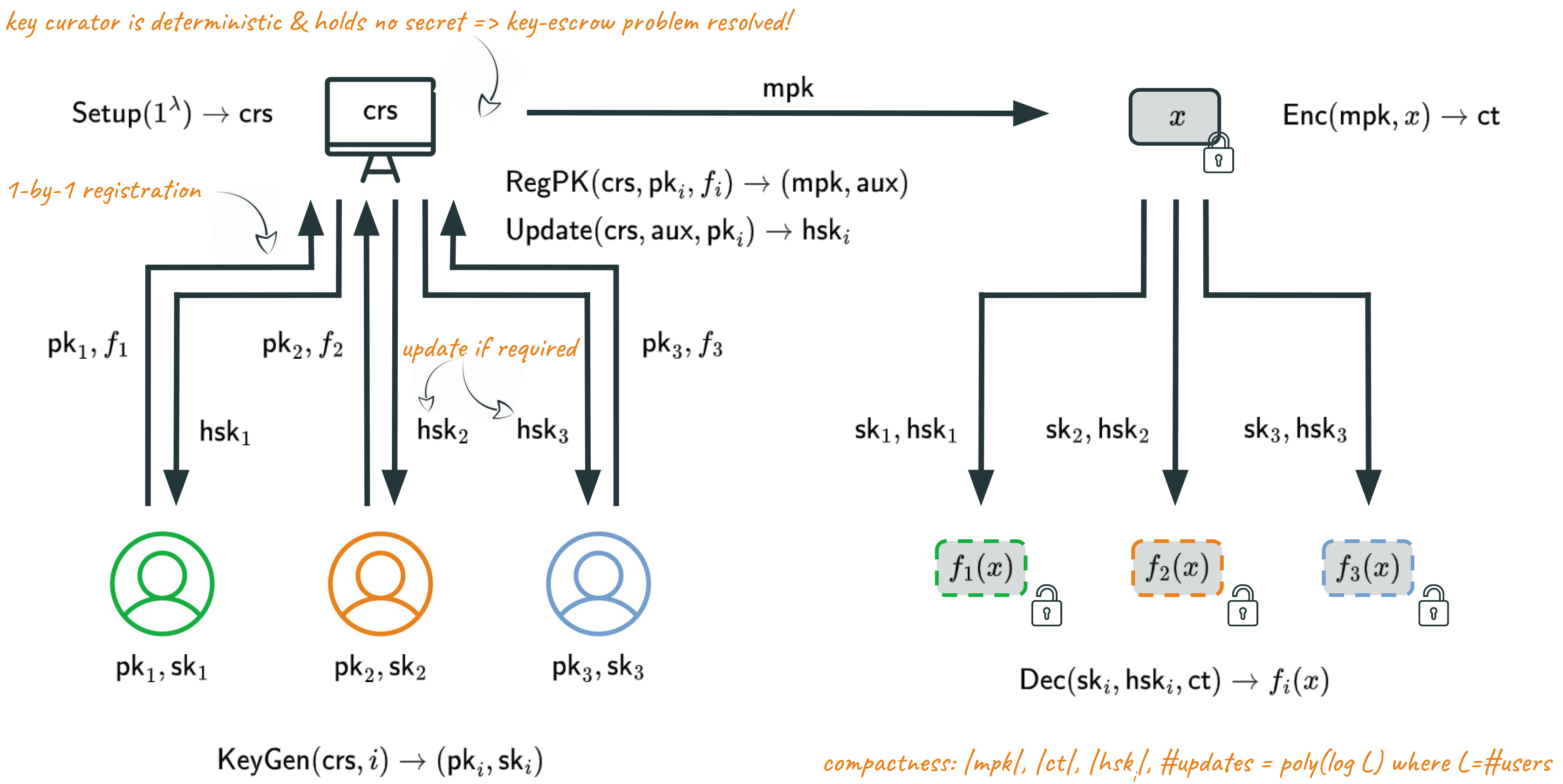
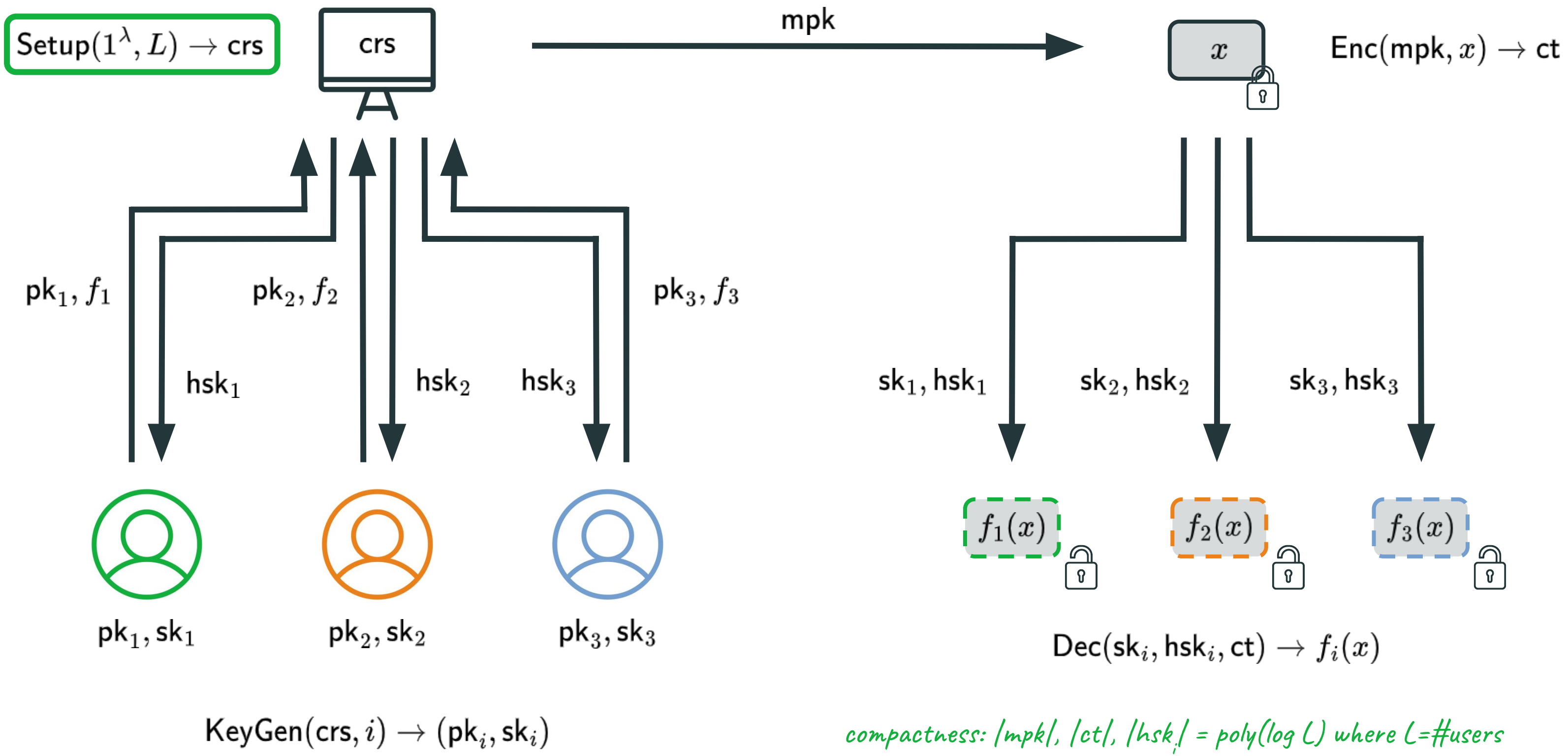


Slotted Registered Functional Encryption (sRFE)



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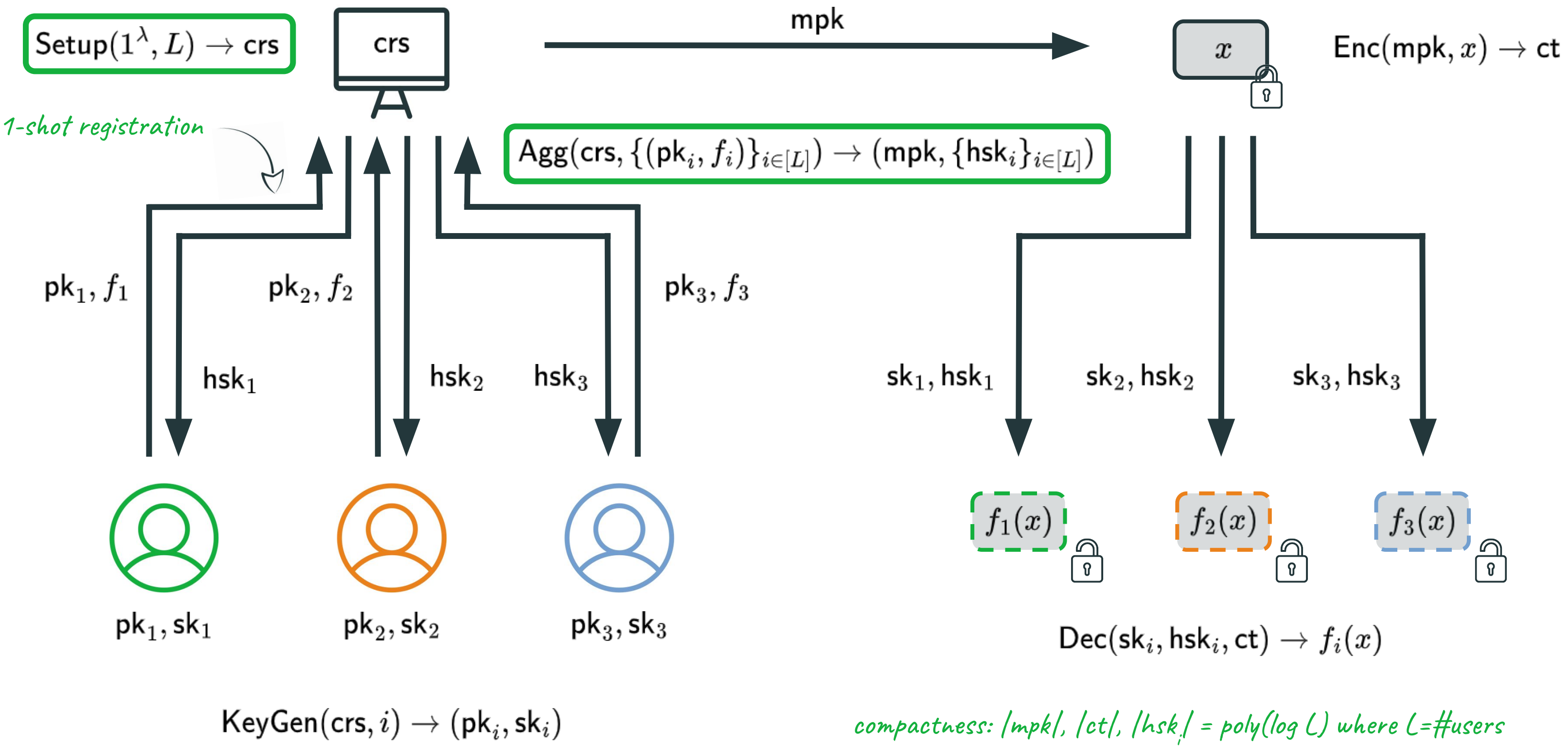

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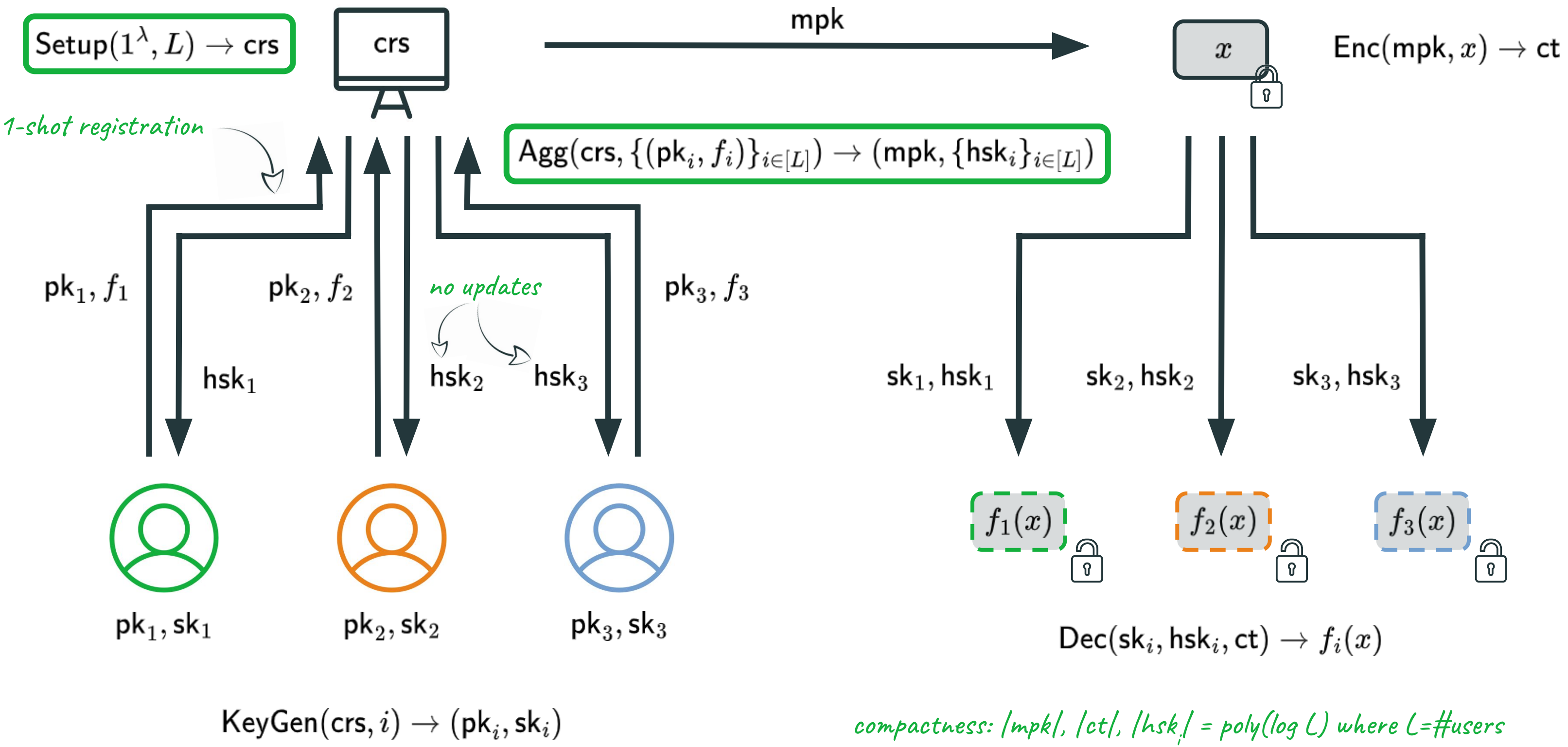
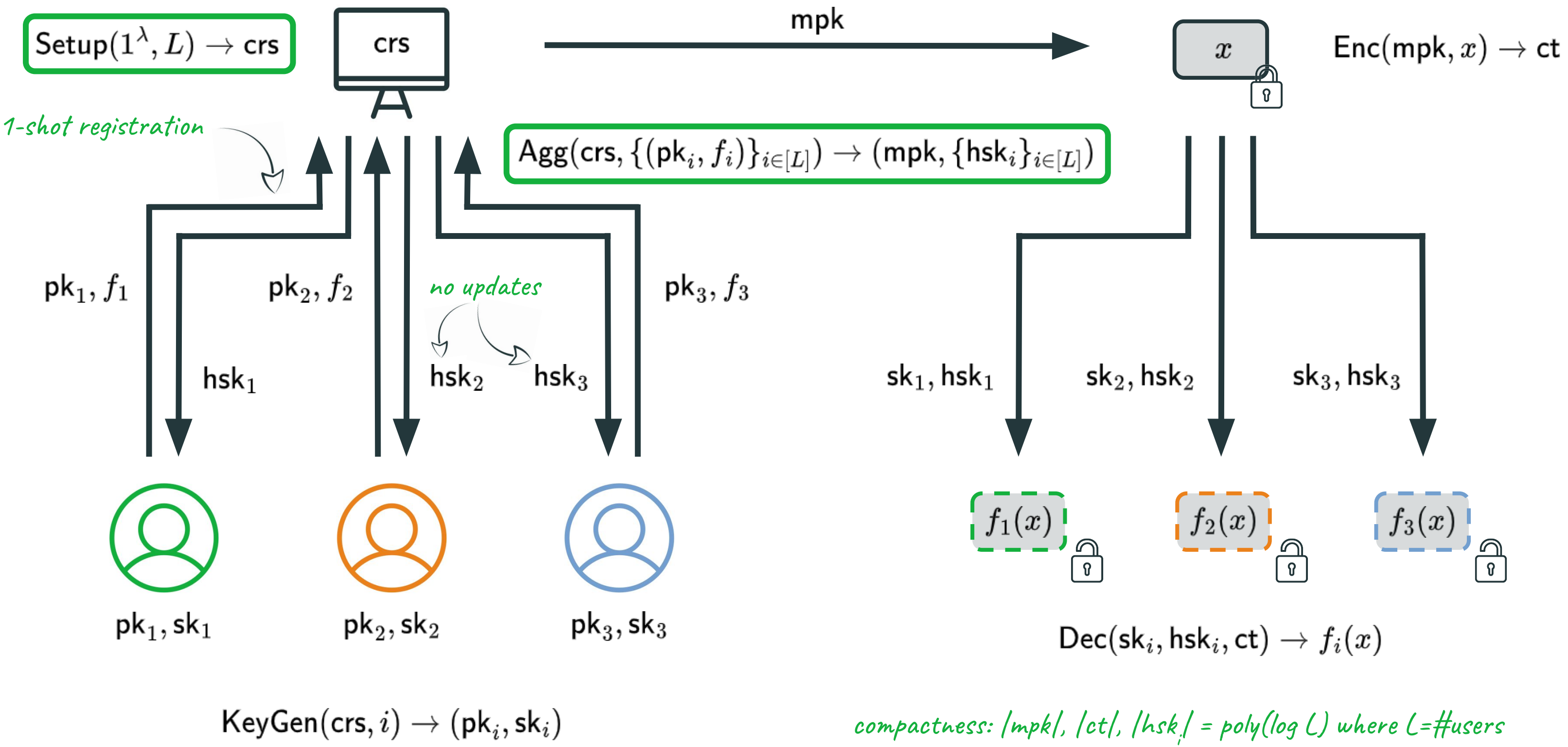


Slotted Registered Functional Encryption (sRFE)

[HLWW23]: sRFE => RFE ("powers-of-two compiler")



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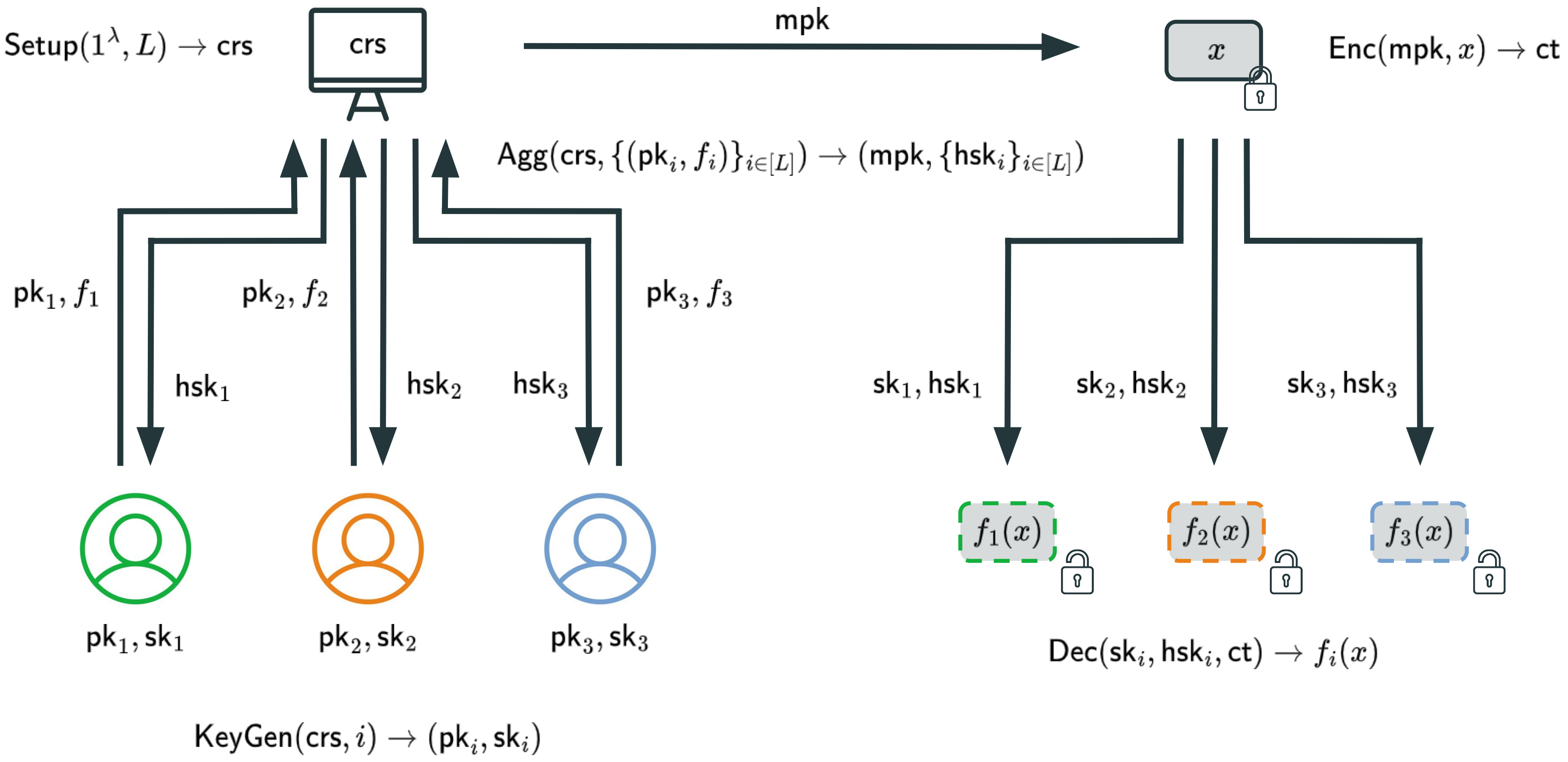

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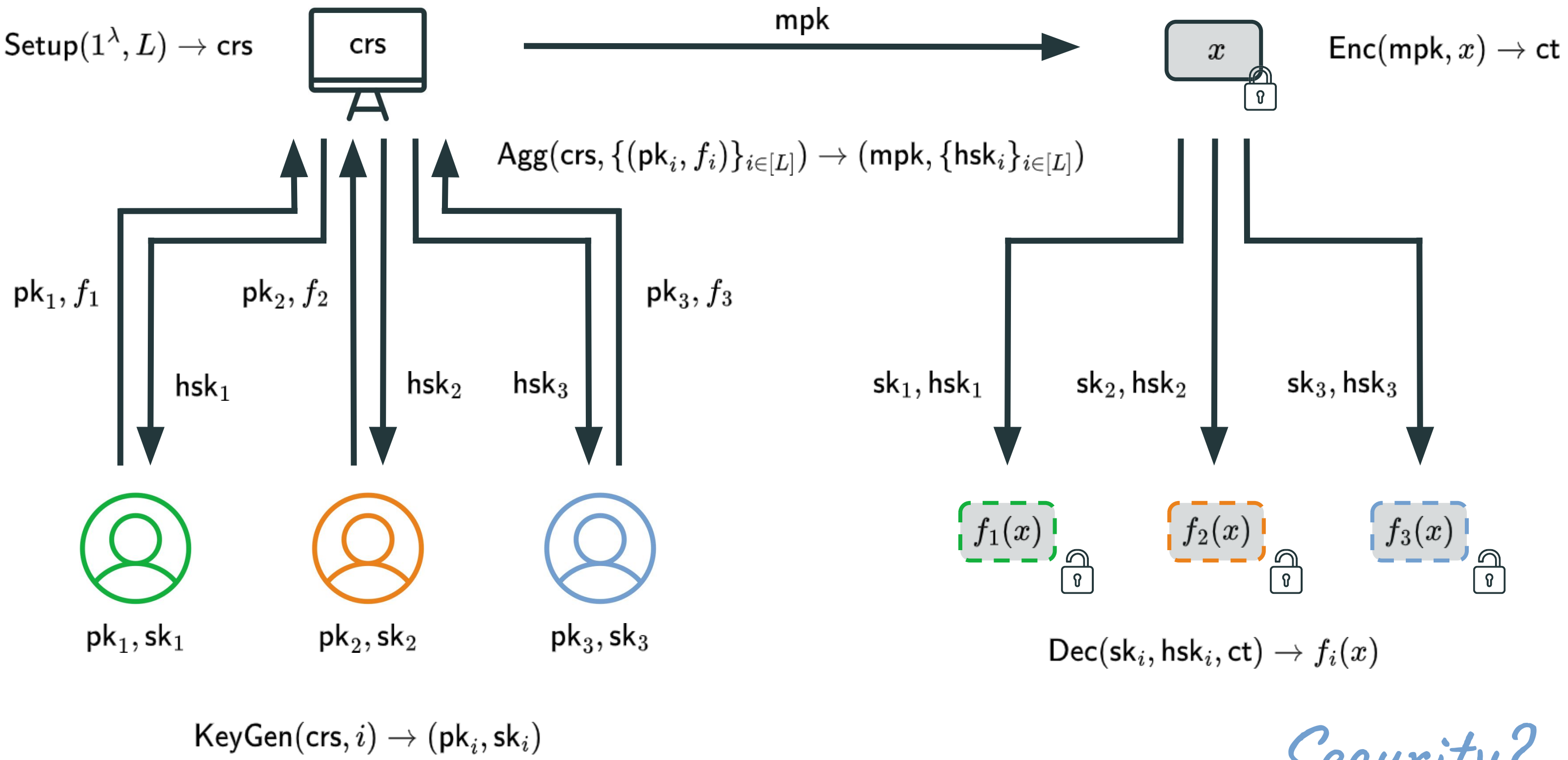

Security?

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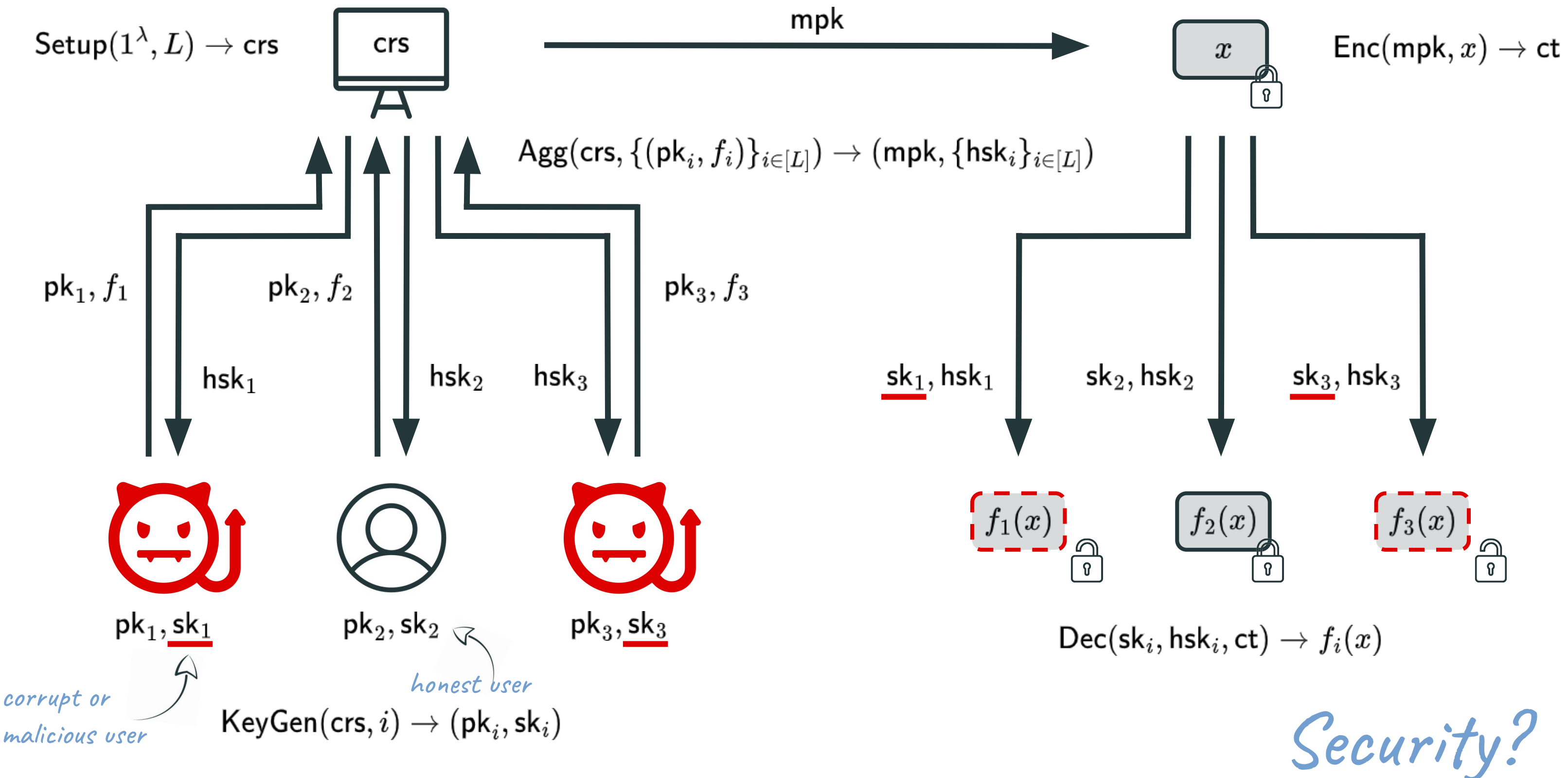

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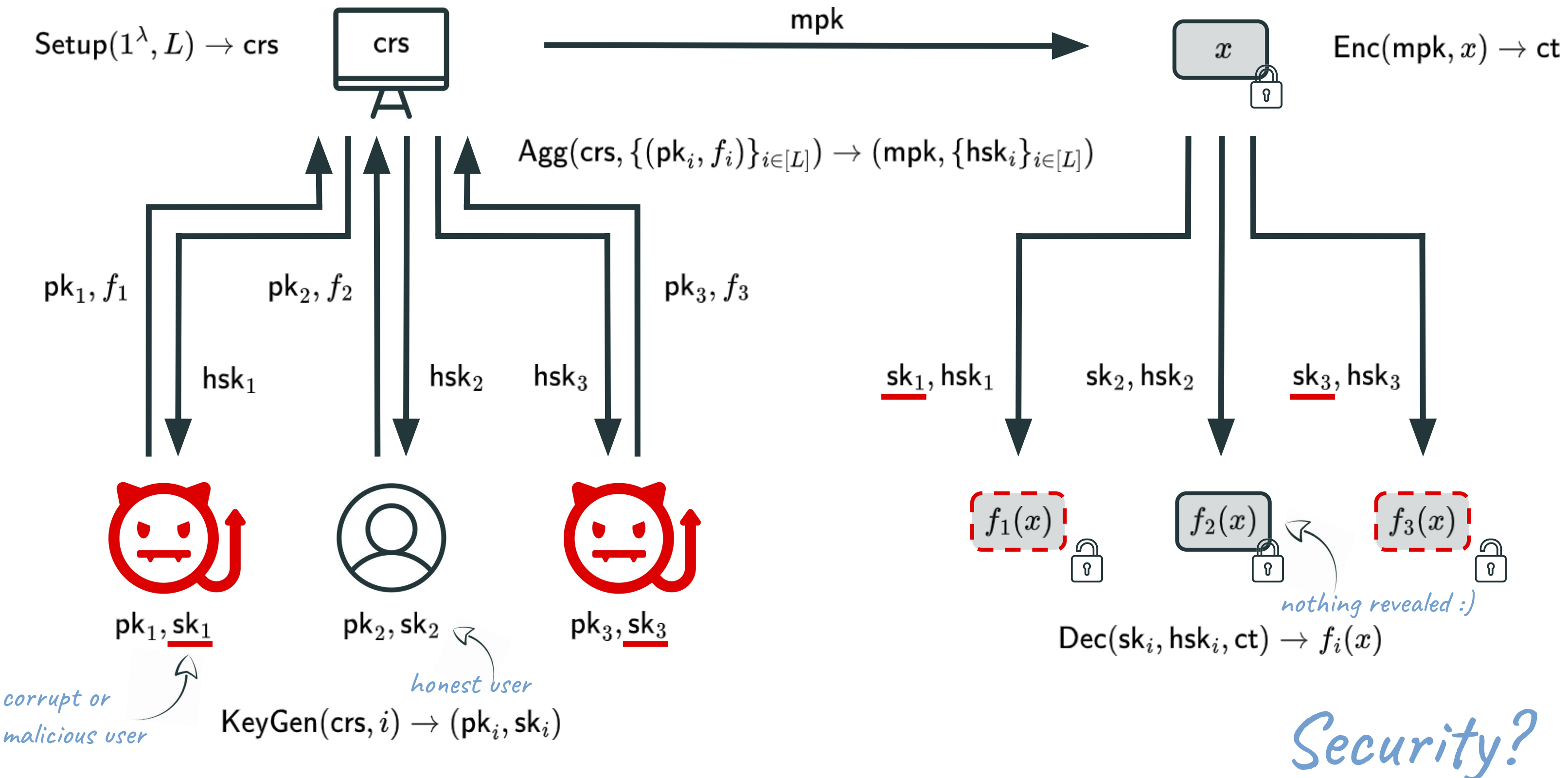

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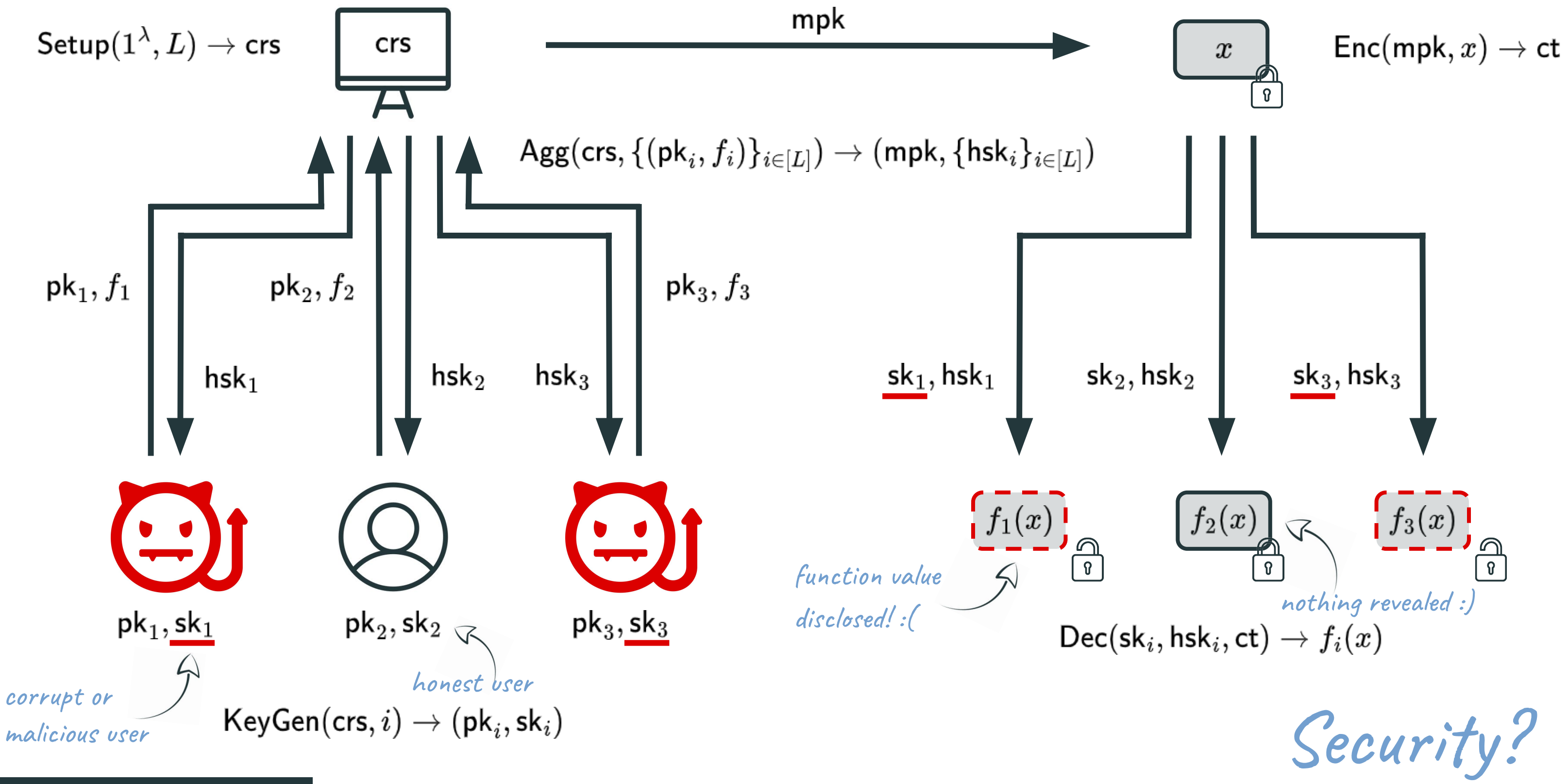
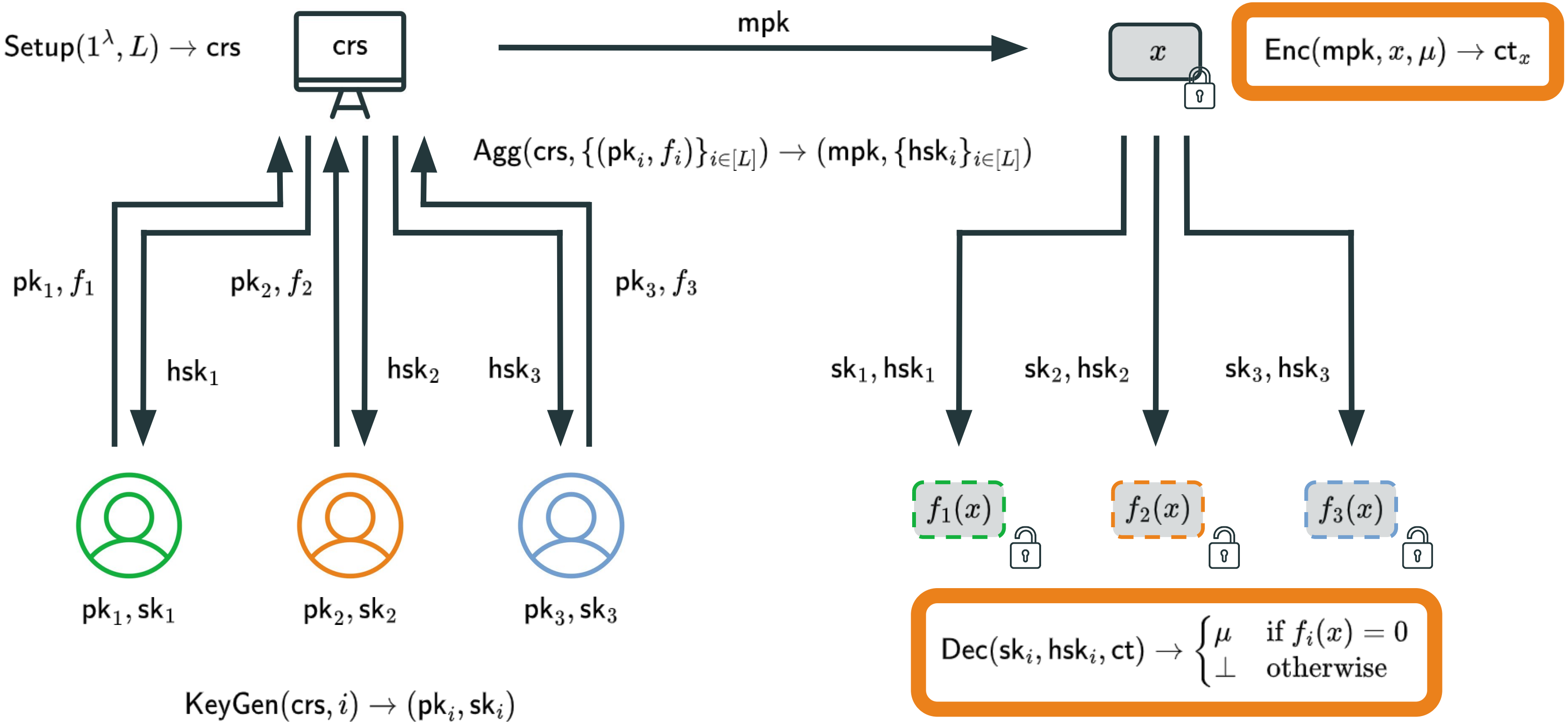
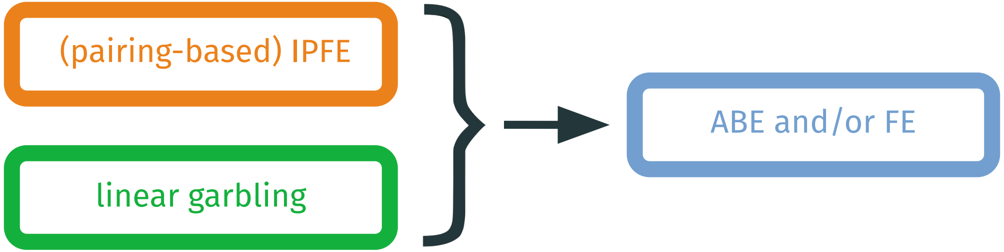


Special Case: Registered ABE



Framework for *Non-Registered* ABE



Framework for *Non-Registered* ABE

(pairing-based) IPFE

linear garbling

$\rightarrow$ ABE and/or FE

Inner-Product Functional Encryption

$\mathbf{u}$ ict($[\mathbf{u}]_1$)

$\mathbf{v}$ isk($[\mathbf{v}]_2$)

$\rightarrow [\mathbf{u}^\top \mathbf{v}]_t$

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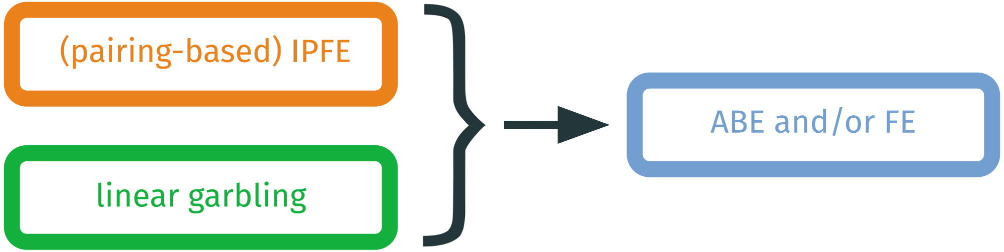

Arithmetic Key Garbling Scheme [EC:LL20]

2-step garbling procedure:

1. given f and secret inputs σ_0, σ_1, output *linear* label functions $L_1(X), \ldots, L_m(X)$ represented by their coefficient vectors $\mathbf{L} = (\mathbf{L}[1], \ldots, \mathbf{L}[m])$
2. given public input $\mathbf{x}$, output label vector $\boldsymbol{\ell} = \mathbf{x} \cdot \mathbf{L} = (L_1(\mathbf{x}), \ldots, L_m(\mathbf{x}))$

evaluation: given $f, \mathbf{x}, \boldsymbol{\ell}$, output $d = \sigma_1 f(\mathbf{x}) + \sigma_0$

simulation: given $f, \mathbf{x}, d$, output $\tilde{\ell} \approx \ell$

Framework for *Non-Registered* ABE

$\mathsf{ct}_{\mathbf{x}}$ $\quad \mu \quad \mathbf{x} \quad$ $\mathsf{ict}([(\mu, s, s \cdot \mathbf{x})]_1)$

sk_f $\quad f \quad$ $\mathsf{isk}([(1, \sigma_0, \mathbf{0})]_2), \{\mathsf{isk}([(0, \mathbf{L}[j])]_2)\}_{j \in [m]}$

$$\longrightarrow [\mu + s \cdot \sigma_0]_t, \underline{\{[\mathbf{x} \cdot s \cdot \mathbf{L}[j]]_t\}_{j \in [m]}}$$

Eval(...) -> f(x) · sσ₁ + sσ₀

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Linearity:

- *labels linear in (1) σ's, r, and (2) x*
- *evaluation linear in labels*

Framework for *Non-Registered* ABE

Security
1. security of IPFE
2. DDH
3. security of AKGS

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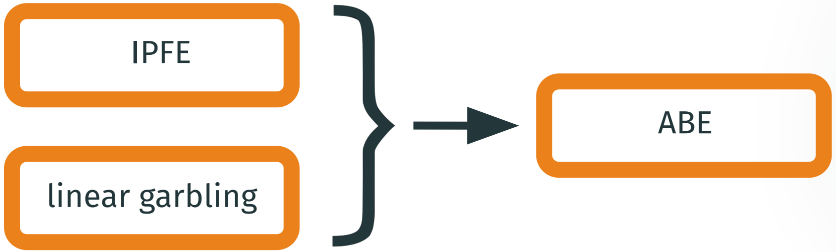

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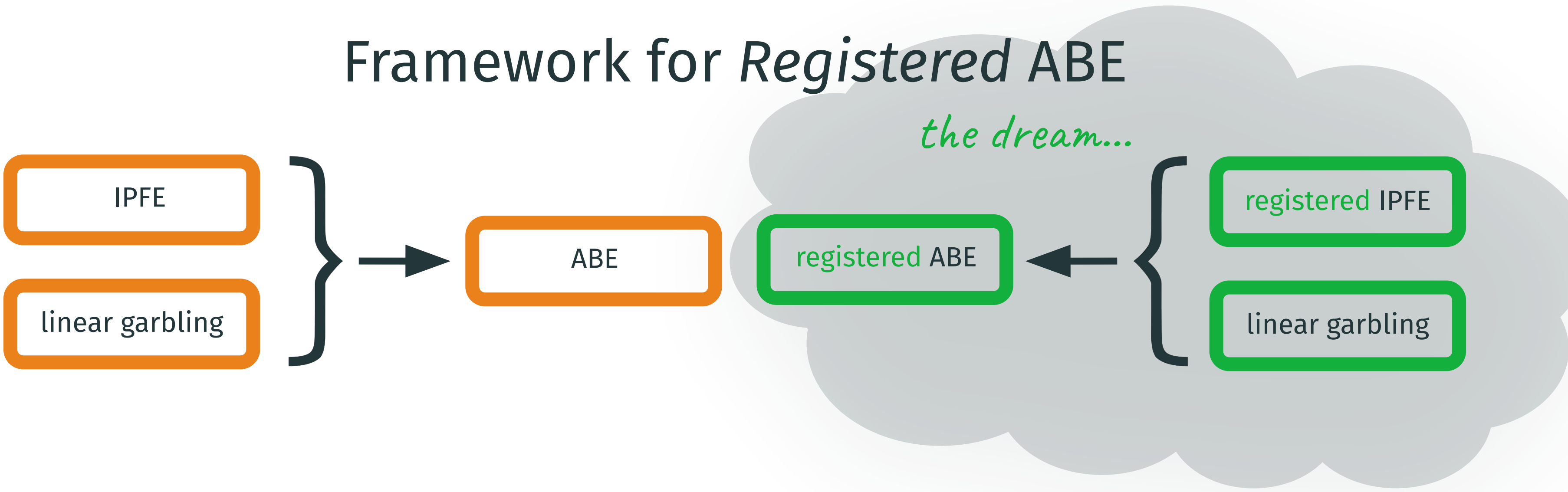

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Linearity to the rescue. $\mathbf{L} = (\sigma_0, \sigma_1, \mathbf{r}) \cdot \widehat{\mathbf{L}}$ -> **offline/online** phase

RFE for Pre-IP (Batch Variant)

- setup: given matrices $\{[\mathbf{P}_i]_2\}_{i \in [L]}$, output **crs**

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Theorem. Assuming bilateral MDDH on pairings, there exists a SIM-secure RFE scheme for Pre-IP.

Proof. (see the paper, based on IND-secure RFE for IP of [EC:ZLZ+24])

How to pick the matrices?

Arithmetic Key Garbling Scheme [EC:LL20]

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Labels are

- linear in secret input and randomness
- linear in public input

$$\boxed{\boldsymbol{\ell} = \big(\mathbf{x} \otimes (\sigma_0, \sigma_1, \mathbf{r})\big) \cdot \widehat{\mathbf{L}}}$$

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$$[\mathbf{u}]_1 = \left[(\mu, s, \mathbf{x} \otimes s)\right]_1 \qquad [\mathbf{P}_i]_2 = \left[\begin{pmatrix} 1 & \\ \sigma_{i,0} & \\ & \mathbf{I}_{|\mathbf{x}|} \otimes (\sigma_{i,0}, \sigma_{i,1}, \mathbf{r}_i) \end{pmatrix}\right]_2 \qquad \mathbf{V}_i = \begin{pmatrix} 1 & \\ & \widehat{\mathbf{L}}_i \end{pmatrix}$$

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Correctness. *RIPFE decryption yields* $[\mathbf{d}_i]_t = [\mathbf{u}\mathbf{P}_i\mathbf{V}_i]_t = \left[\left(\mu + s\sigma_{i,0}, \underline{(\mathbf{x} \otimes (s\sigma_{i,0}, s\sigma_{i,1}, s\mathbf{r}_i)) \cdot \widehat{\mathbf{L}}_i}\right)\right]_t$

Eval(...) -> $f_i(x) \cdot s\sigma_{i,1} + s\sigma_{i,0}$

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Security. *RIPFE leakage is* $[\mathbf{d}_i]_{\mathsf{t}_{1,2}} = [\mathbf{u}\mathbf{P}_i\mathbf{V}_i]_{\mathsf{t}_{1,2}} = \left[\left(\mu + s\overset{\$}{\sigma}_{i,0}, \underline{(\mathbf{x} \otimes (s\overset{\$}{\sigma}_{i,0}, s\overset{\$}{\sigma}_{i,1}, s\overset{\$}{\mathbf{r}}_i)) \cdot \widehat{\mathbf{L}}_i}\right)\right]_{\mathsf{t}_{1,2}}$

indistinguishable from Sim(f, x, d <-$)

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$$\mathbf{V}_i = \begin{pmatrix} 1 & \\ & \widehat{\mathbf{L}}_i \end{pmatrix}$$

What about Turing machines?

Problem: shape of L and r depends on input length, runtime and space -> only known during encryption :(

$$[\mathbf{d}_i]_{\mathrm{t}} = [\mathbf{u}\mathbf{P}_i\mathbf{V}_i]_{\mathrm{t}} = \left[\left(\mu + s\sigma_{i,0}, (\mathbf{x} \otimes (s\sigma_{i,0}, s\sigma_{i,1}, s\mathbf{r}_i)) \cdot \widehat{\mathbf{L}}_i\right)\right]_{\mathrm{t}}$$

Arithmetic Key Garbling Scheme [EC:LL20]

$\mathsf{Garble}(f, \sigma_0, \sigma_1; \mathbf{r}) \to \mathbf{L} = (\mathbf{L}[1], \ldots, \mathbf{L}[m])$

$\mathsf{Eval}(f, \mathbf{x}, \boldsymbol{\ell} := \mathbf{x} \cdot \mathbf{L}) \to d$ s.t. $d = \sigma_1 f(\mathbf{x}) + \sigma_0$

Labels are

- linear in secret input and randomness
- linear in public input

$$\boldsymbol{\ell} = (\mathbf{x} \otimes (\sigma_0, \sigma_1, \mathbf{r})) \cdot \widehat{\mathbf{L}}$$

Arithmetization of TM Computations

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$$\mathbf{T}(\mathbf{x})[c', c] = \begin{cases} 1 & \text{if } c \to_M c' \\ 0 & \text{otherwise} \end{cases}$$

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*... so we only need to garble **matrix multiplication***

Arithmetic Key Garbling for Logspace TMs [EC:LL20]

- garbling: sample $\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_T \leftarrow_\$ \mathbb{Z}_p^{\mathcal{C}}$ and output label functions

$$\begin{aligned} L_{\mathsf{init}}(\mathbf{x}) &= \sigma_0 + \mathbf{r}_0 \cdot \mathbf{e}_{c_0}^\top \\ L_t(\mathbf{x}) &= -\mathbf{r}_{t-1} + \mathbf{r}_t \cdot \mathbf{T}(\mathbf{x}) \\ L_{T+1}(\mathbf{x}) &= -\mathbf{r}_T + \sigma_1 \cdot (\mathbf{1} \otimes \mathbf{y}_{\mathsf{acc}}) \end{aligned}$$

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u V_i P_i

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✔ ✔

- **idea:** set $\mathbf{r}_t = \mathbf{r}_{\mathsf{x},t} \otimes \mathbf{r}_{\mathsf{f}}$ where $\mathbf{r}_{\mathsf{x},t} \leftarrow_\$ \mathbb{Z}_p^{[N]\times[S]\times\{0,1\}^S}$ and $\mathbf{r}_{\mathsf{f}} \leftarrow_\$ \mathbb{Z}_p^{[Q]}$
- **decomposition of the labels:**

$$L_t[\underline{c}, q](\mathbf{x}) = -(\mathbf{r}_{\mathsf{x},t-1} \otimes \mathbf{r}_{\mathsf{f}})[\underline{c}, q] + \big((\mathbf{r}_{\mathsf{x},t} \otimes \mathbf{r}_{\mathsf{f}}) \cdot \mathbf{T}(\mathbf{x})\big)[\underline{c}, q] = -\overset{u}{\mathbf{r}_{\mathsf{x},t-1}[\underline{c}]} \cdot \underset{P_i}{\mathbf{r}_{\mathsf{f}}[q]} + \big((\mathbf{r}_{\mathsf{x},t} \otimes \mathbf{r}_{\mathsf{f}}) \cdot \mathbf{T}(\mathbf{x})\big)[\underline{c}, q]$$

Decomposition of the Labels

- **garbling:** sample $\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_T \leftarrow_\$ \mathbb{Z}_p^{\mathcal{C}}$ and output label functions

$\mathcal{C} = [N] \times [S] \times \{0,1\}^S \times [Q]$

$$\begin{aligned} L_{\text{init}}(\mathbf{x}) &= \sigma_0 + \mathbf{r}_0 \cdot \mathbf{e}_{c_0}^\top & ✔ \\ L_t(\mathbf{x}) &= -\mathbf{r}_{t-1} + \mathbf{r}_t \cdot \mathbf{T}(\mathbf{x}) & ✘ \\ L_{T+1}(\mathbf{x}) &= -\mathbf{r}_T + \sigma_1 \cdot (\mathbf{1} \otimes \mathbf{y}_{\text{acc}}) & ✔ \end{aligned}$$

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Block Structure of Transition Matrix

- transition matrix $\mathbf{T}(\mathbf{x})[c', c] = \begin{cases} 1 & \text{if } \delta(q, \mathbf{x}[k], \mathbf{w}[j]) = (q', \mathbf{w}'[j], k' - k, j' - j),\ \mathbf{w}[\neq j] = \mathbf{w}'[\neq j] \\ 0 & \text{otherwise} \end{cases}$

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block column $(\underline{c} = (k, j, w), \_)$

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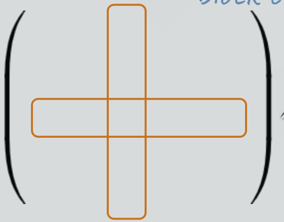
$\mathbf{T}(\mathbf{x}) =$ 

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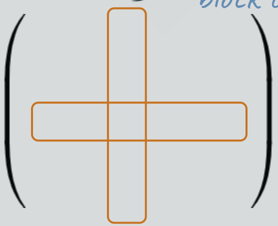

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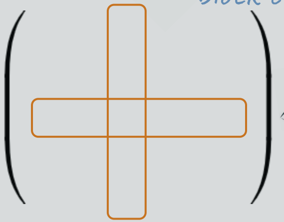
$\mathbf{T}(\mathbf{x}) =$ 

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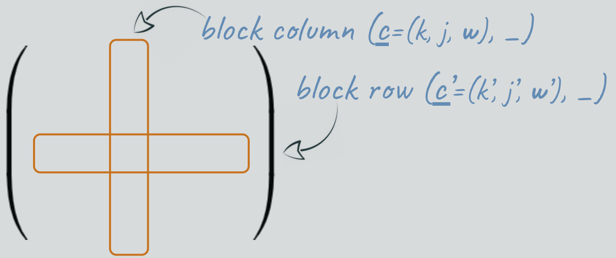
$\mathbf{T}(\mathbf{x}) =$ 

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Generalization to RFE

- so far, we used $\boldsymbol{\sigma_0}$ as a pad for (a fixed message) $\boldsymbol{\mu}$ and $\boldsymbol{\sigma_1}$ as a masking term

Arithmetic Key Garbling Scheme [EC:LL20]

$\mathsf{Garble}(f, \sigma_0, \sigma_1; \mathbf{r}) \to \mathbf{L} = (\mathbf{L}[1], \ldots, \mathbf{L}[m])$

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Thank you!!! :)